J. Fluid Mech. (1991), *vol.* 226, *pp.* 591–624
Printed in Great Britain

The breakdown of the linearized theory and the role of quadrupole sources in transonic rotor acoustics

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(Received 7 September 1989 and in revised form 24 April 1990)

The retarded Green's function for the linearized version of the equation of the mixed type governing the potential flow around a rotating helicopter blade or a propeller (with no forward motion) is derived and is shown to constitute the unifying feature of the various existing approaches to rotor acoustics. This Green's function is then used to pinpoint the singularity predicted by the linearized theory of rotor acoustics which signals its experimentally confirmed breakdown in the transonic regime: the gradient of the near-field sound amplitude, associated with a linear flow which is steady in the blade-fixed rotating frame, diverges on the sonic cylinder at the dividing boundary between the subsonic and supersonic regions of the flow. From the point of view of the equivalent Cauchy problem for the homogeneous wave equation, this singularity is caused by the imposition of entirely non-characteristic initial data on a space–time hypersurface which, at its points of intersection with the sonic cylinder, is locally characteristic. It also emerges from the analysis presented that the acoustic discontinuities detected in the far zone are generated by the quadrupole source term in the Ffowcs Williams–Hawkings equation and that the impulsive noise resulting from these discontinuities would be removed if the flow in the transonic region were to be rendered unsteady (as viewed from the blade-fixed rotating frame).

1. Introduction

Experimental data on the sound generated by a rotating blade whose tip moves at a supersonic speed have been available since the early works of Bryan (1920) and Hilton (1939); nevertheless, many aspects of the existing data on transonic and supersonic rotor acoustics still remain unexplained. As first emphasized by Lilley *et al.* (1953), certain features of these data can already be understood on the basis of the radiative properties of a circularly moving supersonic point source (cf. Lowson 1965; Lowson & Jupe 1974). Historically, however, the theoretical framework of rotor acoustics has developed mainly along the lines of Lighthill's acoustic analogy (Lighthill 1952). When applied to the radiation from a moving body, this approach leads to an inhomogeneous wave equation for the sound amplitude which has three source terms: a monopole term arising from the thickness of the body; a dipole term involving the loading, i.e. the pressure force, on the surface of the body; and a quadrupole term which encompasses all the remaining nonlinear effects (Ffowcs Williams & Hawkings 1969). The formal solution of this wave equation, by means of its retarded Green's function for unbounded space, is known as the Ffowcs Williams–Hawkings equation and constitutes the point of departure for most theoretical studies on the subject (see e.g. Farassat 1975; Hanson 1983).

A knowledge of the exact values of the various source terms in the Ffowcs

Williams–Hawkings equation requires knowledge of the flow field and so of the solution itself. However, in the linearized regime of the theory, in addition to the monopole term that is specified by the shape and the velocity of the rotating blade, the dipole term is known to within the zeroth order in the perturbation quantities, and the quadrupole term which is of the second order can be ignored. For this reason, the earlier calculations in the literature are based primarily on the linearized theory and are concerned only with thickness and loading noise. The agreement between the results of these calculations and the experimental data on helicopter rotors is good only as long as the tip Mach number of the rotating blade lies well below unity; striking discrepancies show up, however, as this Mach number approaches and exceeds 0.9 (Schmitz & Boxwell 1976; Schmitz, Boxwell & Vause 1977; Yu, Caradonna & Schmitz 1978; Boxwell, Yu & Schmitz 1979; Schmitz & Yu 1981). In particular, the flow in the vicinity of the tip of the blade develops shocks (Kittleson 1983) which at higher Mach numbers cease to be local and propagate into the radiation zone in the form of acoustic discontinuities (Schmitz & Yu 1986).

The shock discontinuities observed in the near field are nonlinear features which do in fact emerge from the numerical integration of the nonlinear equation of potential flow in the transonic regime (Caradonna & Isom 1976). A potential flow that is steady in the blade-fixed rotating frame is governed by a partial differential equation which undergoes a change in type – from elliptic to hyperbolic – across the surface where the fluid velocity (in the rotating frame) equals the local sound velocity. The numerical solutions of this mixed equation for appropriate boundary conditions entail the same shock discontinuities in the transonic regime as those observed in the region surrounding the blade tip (see also Sankar & Prichard 1985; Strawn & Caradonna 1986). When the near-field flow obtained from the nonlinear potential-flow equation is used for calculating the dipole and quadrupole source terms in the Ffowcs Williams–Hawkings equation, the far-field nonlinear features – such as the progressive distortion of the waveform with increasing Mach number and the eventual formation of an acoustic discontinuity – are also accounted for by the solution of this equation in the radiation zone (Hanson & Fink 1979; Schmitz & Yu 1986).

Were it not for the considerable body of experimental evidence that shows the breakdown of the linearized theory and for the corresponding results of the nonlinear theory that agree with the data, we would have had no reason to suspect the breakdown of the linearized theory in the transonic regime on the basis of only the earlier linear calculations. Attempts to bridge this logical gap in the development of the subject have already been made in the literature by looking either for singularities (Tam 1983; Myers & Farassat 1987) or for possible missing sources (Farassat & Martin 1983) in the linearized theory, but so far with no success. It is an acknowledged fact that transonic flow is generically nonlinear and unsteady even when produced by small-amplitude disturbances (see Moulden 1984). In other known transonic flows this nonlinearity manifests itself in the singular character of the perturbation problem: a solution of the linearized equations of the flow which is valid in either the subsonic or the supersonic regime becomes singular in the transonic limit (see Cole & Cook 1986, ch. 2). In fact this is not a feature only of transonic flow. A singular perturbation problem arises also in other areas of fluid mechanics whenever the partial differential equation governing the flow is of the mixed type; a ship moving in shallow water, for instance, experiences unexpectedly strong forces when its speed equals the propagation speed of the surface gravity waves that it generates (Tuck 1966; Mei & Choi 1987).

The transonic flow in the blade-fixed coordinate system of a helicopter rotor or a propeller is no exception. In this paper we shall see that in rotor acoustics, too, the linearized theory predicts a physically unacceptable singularity in the transonic regime which signals its breakdown. The sound amplitude of a circularly moving point source which was considered in the earlier studies (Lilley *et al.* 1953; Lowson 1965; Lowson & Jupe 1974) constitutes the Green's function for the linearized version of the potential-flow equation of the mixed type studied by Caradonna & Isom (1976) and various other authors (Sankar & Prichard 1985; Strawn & Caradonna 1986). This same Green's function is moreover the kernel appearing in the quasi-steady form of the Ffowcs Williams–Hawkings equation whose dimensionless part is normally referred to as the Doppler factor. The breakdown of the linearized theory in the transonic regime can be inferred from the structure of this Green's function, which is a common feature of the various approaches to rotor acoustics, without any reference to the specific properties of the source densities that appear in the Ffowcs Williams–Hawkings equation. We shall see, in fact, that the impulsive noise in the far field, which is generated as a consequence of this breakdown and the subsequent formation of shocks in the near field, is also a basic phenomenon which can be understood quite simply in terms of the propagation of discontinuous Cauchy data (representing the near-field discontinuities in the distribution of the quadrupole sources) along the characteristic surfaces of the linear potential-flow equation.

The paper begins with the derivation of the above Green's function in both the time and the frequency domains and first discusses the distinction between the various forms of the Ffowcs Williams–Hawkings equation in the supersonic regime (§2). The singularity structure of this equation appears to undergo a change according to whether the sound amplitude of an extended moving source is regarded as the superposition of the sound amplitudes of the moving infinitesimal volume elements that constitute it, or as the sound amplitude of a stationary source whose constituent parts have the same densities as those of the actual source at the retarded times. The two descriptions are related by a transformation of integration variables in the Ffowcs Williams–Hawkings equation that has a singular Jacobian proportional to the Green's function in question. Provided that the singularity of this Jacobian is handled properly, however, the breakdown of the linearized theory at the sonic cylinder follows from both descriptions (Appendix B).

Section 3 is devoted to a detailed account of the singularity structure of the Green's function in the supersonic regime. The caustic representing the envelope of the spherical wave fronts emanating from a circularly moving point source – which is a surface composed of two sheets that meet along a cusp curve (figure 2) – will be compared and contrasted with the Mach cone of a rectilinearly moving point source, and its role in the calculation of the sound amplitude of an extended source, as the inverted bifurcation surface of a catastrophe, will be discussed. The source points on the bifurcation surface approach the observation point at the speed of sound and so constitute the loci of the singularities of the Green's function; amongst these the points on the cusp curve, which at the same time are approaching the observation point with zero acceleration, represent singularities that are stronger than those appearing in the rectilinear case. In the limit where the actual speed of the point source approaches the speed of sound, the whole bifurcation surface collapses onto its cusp curve and so gives rise to yet a higher-order singularity in the Green's function.

In §4 we shall see that although all these singularities of the Green's function are integrable and so, according to the linearized theory, the sound amplitude due to an extended source is itself everywhere finite, the radial component of the gradient of the sound amplitude predicted by the Ffowcs Williams–Hawkings equation diverges as the observation point approaches the sonic cylinder from outside. This sudden and infinitely large change in the gradient of the sound amplitude that occurs as the radial position of the observation point (playing the role of the control parameter of a catastrophe) crosses the sonic cylinder, is a reflection of the fact that the strong sound fields generated by the supersonically moving volume elements of an extended source at their caustics are entirely confined to the supersonic region, and cannot *propagate* across the sonic cylinder to reach an observer who is located just inside this surface. Not only do the waves emitted in the supersonic region interfere constructively to form caustics but, because the equation governing the sound field of a curvilinearly moving source is of the mixed type, these caustics – which constitute the ray conoid of the governing equation – cannot penetrate the subsonic region where the equation is elliptic.

The speed-of-sound catastrophe discussed in §4 arises from those contributions towards the values of the integrals in the Ffowcs Williams–Hawkings equation which are made by the source elements in the immediate vicinity of the singularities of the Green's function. Unless we exclude the singularities in the integrands of the integrals in question from the domain of integration before differentiating these integrals, the relevant contributions which come from the boundaries of the excluded regions will not appear in the expression for the gradient of the sound amplitude. The physically observable results of a calculation are of course the same whether they are obtained by means of the theory of classical or of generalized functions; however, the order in which the two operations of differentiation and integration are performed may be interchanged, as is commonly done in the literature on rotor acoustics, only if the theory of generalized functions is applied consistently. This, and the fact that it is the near-field rather than the extensively studied far-field sound amplitude whose gradient diverges, explain why the earlier calculations in the literature fail to pinpoint the singularity which is responsible for the breakdown of the linearized theory.

In §5 we formulate the Cauchy problem for the homogeneous wave equation that is equivalent to the inhomogeneous problem discussed in §4, and show that the initial data for this problem have to be prescribed on a hypersurface in space–time which is null, i.e. characteristic, at points where it intersects the sonic cylinder. It is well known, however, that unless the data are also characteristic at points where the initial hypersurface becomes characteristic, the Cauchy problem cannot have a solution whose derivatives are regular (see Bleistein 1984). So, from this alternative point of view, the radial gradient of the solution discussed in §4 is singular because the Cauchy data required to duplicate the type of source in question are not characteristic at the sonic cylinder. The locus of the singularity is in the present case the envelope of those characteristics of the wave equation which are stationary in the rotating frame.

In §6 we specify the surface of parabolic degeneracy of the nonlinear potential-flow equation – on which this equation undergoes a change in type – and, on the basis of the location of this surface relative to that of the linear sonic cylinder, we explain why there is a breakdown in the linearized theory before the tip Mach number of a blade attains the value 1. Since the velocity potential and the velocity field on any surface located just outside the surface of parabolic degeneracy uniquely specify the flow in the entire volume where the potential-flow equation is hyperbolic (Isom, Purcell & Strawn 1987; Schulten 1988), the acoustic radiation field is in the present case influenced by the flow in the near zone only to the same extent that the conditions at the base of the supersonic region are influenced; these conditions constitute the Cauchy data for the quasi-steady potential equation in its domain of hyperbolicity and propagate along the bicharacteristics of this equation into the far zone. The acoustic discontinuities observed in the radiation field occur across the characteristic surfaces of the potential-flow equation (which are stationary in the blade-fixed rotating frame) and arise from the discontinuous Cauchy data at the base of the supersonic region.

In §7 we briefly remark on the practical implications of the paper's results and discuss the relationship of the singularity obtained here with those which have already been noted in the literature. These earlier singularities invariably arise from sources (such as those analysed in Appendix C) which are themselves singular.

Most of the mathematical results of the present paper pertain to general properties of the retarded potential and have already been derived in Ardavan (1989) which will be referred to as Paper I. Here we give a descriptive account of the previous results and only present those calculations which either do not appear in Paper I or arise in the specific context of rotor acoustics.

2. The theoretical framework of rotor acoustics

The generation of sound by a rigid body in arbitrary motion through a fluid is governed by the following wave equation, known as the Ffowcs Williams–Hawkings equation (Ffowcs Williams & Hawkings 1969), which is an exact consequence of the conservation laws for mass and momentum:

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \nabla^2 \rho = s, \tag{1}$$

with

$$s = \frac{\partial^2}{\partial x_i \partial x_j}[T_{ij}\,\theta(f)] - \frac{\partial}{\partial x_i}\left[p_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right] + \frac{\partial}{\partial t}\left[\rho_0 v_i \frac{\partial f}{\partial x_i}\delta(f)\right], \tag{2}$$

where ρ and p_{ij} are the deviations of the density and the stress tensor of the ambient fluid from their mean values ρ_0 and $p_0\delta_{ij}$, the constant c is the mean value of the speed of sound, $\boldsymbol{v}$ is the local velocity with which the surface of the body encroaches on the fluid, and $f(\boldsymbol{x}, t) = 0$ defines the surface of the moving body with $f < 0$ inside and $f > 0$ outside the body; the tensor $T_{ij} = (\rho_0 + \rho)\,u_i u_j + p_{ij} - \rho c^2 \delta_{ij}$, in which $\boldsymbol{u}$ stands for the fluid velocity, is Lighthill's stress tensor. In these expressions $\boldsymbol{x}$ is the position vector, t is time, $\theta(f)$, $\delta(f)$ and δ_{ij} are the Heaviside step function and the Dirac and the Kronecker delta functions, respectively, and the indices i and j designate Cartesian components of tensors and are to be summed over the values 1, 2, 3 when repeated.

Equation (1) is a nonlinear equation because, even though the source term in this equation is known to within the zeroth order in the perturbation quantities, a knowledge of the exact value of s requires knowledge of the flow field and so of the solution itself. Nevertheless, it is possible to use the following retarded Green's function for the linear wave equation in unbounded space:

$$G = \delta(t_{\mathrm{P}} - t - R/c)/R \tag{3}$$

to rewrite (1) formally as

$$\rho(\boldsymbol{x}_{\mathrm{P}}, t_{\mathrm{P}}) = \frac{1}{4\pi c^2}\int_V \mathrm{d}^3x \int_{-\infty}^{+\infty} \mathrm{d}t\, s(\boldsymbol{x}, t)\, G(\boldsymbol{x}, t; \boldsymbol{x}_{\mathrm{P}}, t_{\mathrm{P}}), \tag{4}$$

where $(\boldsymbol{x}_P, t_P)$ and $(\boldsymbol{x}, t)$ denote the space-time coordinates of the observation and the source points, respectively, R stands for $|\boldsymbol{x}-\boldsymbol{x}_P|$, and V extends over all space.

When the body in question is a hovering blade with no forward velocity and so its motion consists of a rigid rotation about a fixed axis with a constant angular frequency ω, only the quasi-steady sources of sound, whose strengths do not vary with time in the blade-fixed coordinates, are important (see Hawkings & Lowson 1974), and we are concerned with flow variables which depend on the azimuthal angle φ and the time t as functions of the single variable $\varphi-\omega t$. Thus for rotor acoustics, in which ρ, f and the cylindrical components of $\boldsymbol{u}$, $\boldsymbol{v}$ and p_{ij} possess the symmetry $\partial/\partial t+\omega\,\partial/\partial\varphi = 0$, the source term of (1) has the form

$$\left.\begin{aligned} s(r,\varphi,z,t) &= s(r,z,\hat{\varphi}),\\ \hat{\varphi} &\equiv \varphi-\omega t, \end{aligned}\right\} \tag{5}$$

where the z-axis of the cylindrical polar coordinates (r,φ,z) is defined by the axis of rotation. (This can be seen by transforming the partial derivatives with respect to Cartesian coordinates that appear in (2) into covariant derivatives with respect to cylindrical coordinates, and noting that the process of differentiation does not introduce any dependence on the individual variables φ or t.)

If we now insert (5) in (4), write R in its cylindrical form

$$R = [(z-z_P)^2+r^2+r_P^2-2rr_P\cos(\varphi-\varphi_P)]^{\frac{1}{2}}, \tag{6}$$

and change the variables of integration from (r,φ,z,t) to $(r,\varphi,z,\hat{\varphi})$, the sound amplitude becomes

$$\rho(r_P,\hat{\varphi}_P,z_P) = \frac{1}{4\pi c^2}\int_{\hat{V}} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\hat{\varphi}\,s(r,\hat{\varphi},z)\,G_0, \tag{7}$$

where

$$G_0 = \oint \mathrm{d}\varphi\,\frac{\delta(\varphi-\hat{\varphi}-\omega t_P+R\omega/c)}{R}, \tag{8}$$

and $\hat{V}$ extends over the support of the generalized function $s(r,\hat{\varphi},z)$. Performing the integration over φ in (8), we obtain

$$G_0 = \sum_{\varphi=\varphi_j}\frac{1}{R|1-M_R|}, \tag{9}$$

in which

$$M_R = \frac{rr_P\omega}{cR}\sin(\varphi_P-\varphi) = \frac{r\omega}{c}\hat{\boldsymbol{e}}_\varphi\cdot\frac{\boldsymbol{x}_P-\boldsymbol{x}}{|\boldsymbol{x}_P-\boldsymbol{x}|} \tag{10}$$

is the Mach number with which the source point approaches the observation point, and φ_j are those solutions of the functional equation $\varphi-\hat{\varphi}-\omega t_P+R\omega/c = 0$ that lie within the interval of 2π over which the integral is evaluated. ($\hat{\boldsymbol{e}}_\varphi$ is the cylindrical basis vector associated with the azimuthal angle φ.)

The function G_0 is simply the sound amplitude due to a circularly moving point source and represents the Green's function for the quasi-steady wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho}{\partial r}\right)+\frac{\partial^2\rho}{\partial z^2}+\left(\frac{1}{r^2}-\frac{\omega^2}{c^2}\right)\frac{\partial^2\rho}{\partial\hat{\varphi}^2} = -\frac{1}{c^2}s(r,\hat{\varphi},z) \tag{11}$$

which follows from (1) and the symmetry condition

$$\frac{\partial}{\partial t}+\omega\frac{\partial}{\partial \varphi}=0. \tag{12}$$

We have here obtained G_0 from the four-dimensional Green's function G of the wave equation by the method of descent (cf. Courant & Hilbert 1962, p. 205). However, G_0 can also be obtained directly by solving (11), which applies to the lower-dimensional $(r,\hat{\varphi},z)$-space, for the point source

$$s=\frac{4\pi c^2}{r}\delta(r-r_{\mathrm{P}})\,\delta(z-z_{\mathrm{P}})\,\delta(\hat{\varphi}-\hat{\varphi}_{\mathrm{P}}). \tag{13}$$

(Note that the potential that arises from a point source located at $r_{\mathrm{P}}=r$, $z_{\mathrm{P}}=z$, $\hat{\varphi}_{\mathrm{P}}=\hat{\varphi}$ in the $(r_{\mathrm{P}},z_{\mathrm{P}},\hat{\varphi}_{\mathrm{P}})$-space is governed by the same equation as that which governs the potential due to a point source at $r=r_{\mathrm{P}}$, $z=z_{\mathrm{P}}$, $\hat{\varphi}=\hat{\varphi}_{\mathrm{P}}$ in the $(r,z,\hat{\varphi})$-space.)

Fourier-transforming the $\hat{\varphi}$ and z dependences of both sides of (11) for the point source (13) and requiring, in the standard way, that the solution to the resulting ordinary differential equation should be regular at $r=0$, should itself be continuous but have a discontinuous first derivative at $r=r_{\mathrm{P}}$, and should represent an outgoing wave at infinity, we find

$$G_0=\mathrm{i}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\hat{\varphi}_{\mathrm{P}}-\hat{\varphi})}\int_0^{\infty}J_m(\lambda r_<)\,H_m^{(1)}(\lambda r_>)\cos\left[\kappa(z-z_{\mathrm{P}})\right]\mathrm{d}\kappa, \tag{14}$$

where

$$\lambda=\begin{cases}\dfrac{m\omega}{c}\left(1-\dfrac{\kappa^2c^2}{m^2\omega^2}\right)^{\frac{1}{2}}, & \dfrac{\kappa^2c^2}{m^2\omega^2}<1\\[2ex] \mathrm{i}\left(\kappa^2-\dfrac{m^2\omega^2}{c^2}\right)^{\frac{1}{2}}, & \dfrac{\kappa^2c^2}{m^2\omega^2}>1,\end{cases} \tag{15}$$

$r_<(r_>)$ is the smaller (larger) of r_{P} and r, J_m is the Bessel function of order m and $H_m^{(1)}$ is the Hankel function of the first kind and the mth order. This series is identical to that obtained by the Fourier expansion of the right-hand side of (9) (see Ardavan 1984), and confirms that G_0 is in fact the retarded Green's function for (11). (The classical results of Gutin 1936 follow from the far-field version of (14) which appears in Ardavan 1981.)

The differential operator appearing on the left-hand side of (11) is not – like that in the original Ffowcs Williams–Hawkings equation – hyperbolic, but of the mixed type: it is elliptic in the subsonic regime $r\omega<c$ and hyperbolic in the supersonic regime $r\omega>c$. In the same way that the symmetry $\partial/\partial t=0$ turns the wave equation into the lower-dimensional Poisson's equation that is elliptic, so the symmetry $\partial/\partial t+\omega\,\partial/\partial\varphi=0$, which is with respect to time in $r<c/\omega$ and with respect to space in $r>c/\omega$ (Paper I, §6), turns the wave equation into an equation of the mixed type over the $(r,z,\hat{\varphi})$-subspace of space–time. Our using the retarded potential for the derivation of (9) is analogous to descending from the retarded potential to the solution (19) of Poisson's equation and does not mean that the equation we have solved is the ordinary wave equation. The basic equation of linearized rotor acoustics, i.e. (11) for a known s, is both of a different dimension and of a different type from the wave equation.

This fact enables us to distinguish between the different forms of the Ffowcs Williams–Hawkings equation which are obtained from (4) by different choices of the first integration variable (Ffowcs Williams & Hawkings 1969; Hawkings & Lowson 1974; Farassat 1981). In rotor acoustics there is only one of these forms which can also be obtained directly from (11): the form presented in (7) that is obtained when the integration of the delta function in (4) affects a descent onto the $(r, \hat{\varphi}, z)$-space.

If, as is customary, the delta function is integrated over t, then (4) and (5) yield the following expression for the sound amplitude:

$$\rho(r_{\mathrm{P}}, \hat{\varphi}_{\mathrm{P}}, z_{\mathrm{P}}) = \frac{1}{4\pi c^2} \int_V r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\varphi \frac{s(r, z, \varphi - \omega t_{\mathrm{P}} + R\omega/c)}{R}, \tag{16}$$

which is the Duhamel's form of the retarded potential (Courant & Hilbert 1962, p. 202). The integrand in this equation is, in contrast to that in (7), non-singular for $M_R = 1$ because the change of variables $\hat{\varphi} = \varphi - \omega t_{\mathrm{P}} + R\omega/c$, which would have to be made to transform it into the former integrand, has a singular Jacobian equal to the Doppler factor $|1 - M_R|^{-1}$. Since such a transformation is not permissible, unless we exclude the points at which the Jacobian is irregular or zero from the domain of integration, in going from (7) to (16) we have in effect shifted the singularity from the integrand onto the limits of the integral. This point is clearly illustrated by the following example which concerns a simpler rectilinearly moving source.

It is well known (Tolman, Ehrenfest & Podolsky 1931; Dowling & Ffowcs Williams 1983; Ardavan 1984; Cole & Cook 1986) that the sound amplitude at a point inside an extended source which moves at the speed of sound in a straight line, diverges like $\ln t_{\mathrm{P}}$ when the duration t_{P} of the motion tends to infinity. Physically, the diverging contribution towards the sound amplitude comes, in this case, from the constructively interfering waves that are produced by the source elements in the immediate vicinity of the observation point. Mathematically, however, the origins of the singularity appear in different guises depending on whether we use the rectilinear counterpart of (7) – which is given in §II of Ardavan (1984) – or that of (16). When calculated by means of the counterpart of (7), the divergence arises from the singularity structure of the corresponding Green's function, whereas when calculated by means of the counterpart of (16), it arises from the unboundedness of the range of integration. Because for the source elements in the immediate vicinity of the observation point a short interval of observation time corresponds to an infinitely long interval of retarded time, the change of variables that transforms the former integral into the latter at the same time maps what in the comoving frame is a small volume just ahead of the observation point into a source whose extension in the direction of motion is unbounded.

What distinguishes (7) from the other forms of the Ffowcs Williams–Hawkings equation is that in rotor acoustics the source density s is known as a function of the blade-fixed coordinates $(r, z, \hat{\varphi})$. To use the alternative form given in (16), for instance, we need to map the interval in $\hat{\varphi}$ over which s is non-zero into the corresponding support of s in φ. The mapping from $\hat{\varphi}$ to φ is multi-valued in the supersonic regime and, unless the zeros of its Jacobian are handled properly, does not result in an unambiguous integral which we can in fact differentiate without first evaluating. However, we shall see in §4 and Appendix B that, provided that we exclude the zeros of the Jacobian of this mapping from the range of integration prior to differentiating (16) and only proceed to the limit in which the volume of the excluded region shrinks to zero after having completed the calculation, then the singularity in the gradient of the sound amplitude at the sonic cylinder follows from (16), as well as from (7) on which the following analysis is based. (Note that to exclude a certain region from the domain of integration and to proceed to the limit in which the volume of the excluded region shrinks to zero at the end of the calculation is a mathematically permissible step in the evaluation of the derivative of any integral; in cases where there are no net contributions from the boundaries of the excluded region and so this step is spurious, the limiting operation would simply leave the outcome of the calculation unchanged.)

Equations (7) and (16) are obtained by projecting the past light cone of the observer – which constitutes the support of the integrand in (4) – onto the 3-surfaces $\varphi = \text{const.}$ and $\hat{\varphi} = \text{const.}$ of the four-dimensional space $(r, \varphi, z, \hat{\varphi})$, respectively. We may of course project the past light cone of the observer onto any hypersurface in space–time and so obtain a new form of the equation; the new integrand will be related to those in (7) or (16) by a transformation of the integration variables. Thus, the Jacobian of this transformation, which depends on the angle between the original and the new hypersurfaces, can be made arbitrarily singular. However, unless the singularity of the new integrand is integrable, the values of the two integrals will not be the same, i.e. the transformation will not be permissible, no matter how carefully the singularity of its Jacobian is handled.

3. The supersonic regime of the theory

Although the sound field of a circularly moving point source, i.e. the Green's function G_0, enters most theoretical works on rotor acoustics at least implicitly, a study of its properties – comparable in detail to that of the well-known properties of the sound field of a rectilinearly moving point source – has not yet appeared in the literature on this subject. (For earlier discussions of the problem see Hilton 1939; Lilley *et al.* 1953, and Lowson & Jupe 1974; for the mathematical details of the account given in this section see Paper I, §3.) Just as the spherical field wavelets emanating from a rectilinearly moving supersonic point source form a Mach cone at which the sound amplitude is infinite (figure 1), so the envelope of the corresponding wavelets from a circularly moving supersonic point source constitutes a caustic in the $(r_P, \hat{\varphi}_P, z_P)$-space at which the factor $|1 - M_R|$ in the denominator of (9) vanishes and the amplitude G_0 diverges. This caustic begins issuing from the point source in the form of a cone with the same opening angle, $\nu = \arcsin (r\omega/c)^{-1}$, as that of the Mach cone and, after joining a second sheet, eventually develops into a tube-like surface which spirals around the rotation axis (figure 2). The two sheets of the caustic are tangent to one another and so form a cusp where they meet. The resulting cusp is a distorted U-shaped curve whose two segments run along the two sides of the spiralling tube-like surface from where the cone becomes a tube to infinity (see also the figures in Lilley *et al.* 1953; da Costa & Kahn 1985; and Paper I).

When we superpose the sound amplitudes from the volume elements which constitute an extended source, as in the expression for $\rho(r_P, \hat{\varphi}_P, z_P)$ in (7), the coordinates of the observation point $(r_P, \hat{\varphi}_P, z_P)$ are fixed and we are primarily concerned with the behaviour of the Green's function G_0 which appears in the integrand of this equation as a function of the coordinates $(r, \hat{\varphi}, z)$ of the source point. In the $(r, \hat{\varphi}, z)$-space, the singularity of G_0 occurs on a surface which is the mirror image of the caustic, i.e. the reflection of the caustic issuing from the fixed point $(r_P, \hat{\varphi}_P, z_P)$ across the meridional plane passing through its conical apex (see the broken curve in figure 2). The corresponding surface in the rectilinear case is an inverted Mach cone: of the source points comprising the extended source shown in figure 1, the ones which produce an infinite sound amplitude at the observation point P are those which lie on the inverted Mach cone shown by the broken lines. While the

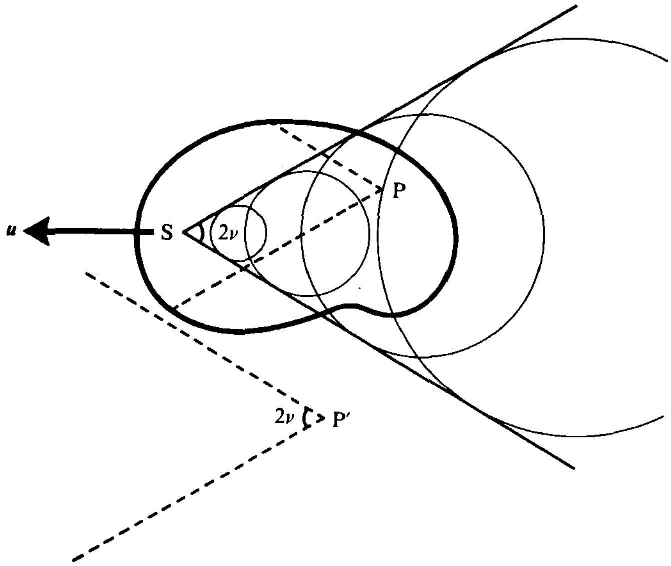

FIGURE 1. The irregular closed curve designates the boundary of an extended source which moves rectilinearly at a constant supersonic velocity $\boldsymbol{u}$. The envelope of the spherical wave fronts emanating from the source point S is the Mach cone associated with this particular volume element of the source and has the opening angle $\nu = \arcsin(c/u)$. The inverted Mach cones, depicted by broken lines, represent the bifurcation surfaces of two arbitrary observation points P and P′.

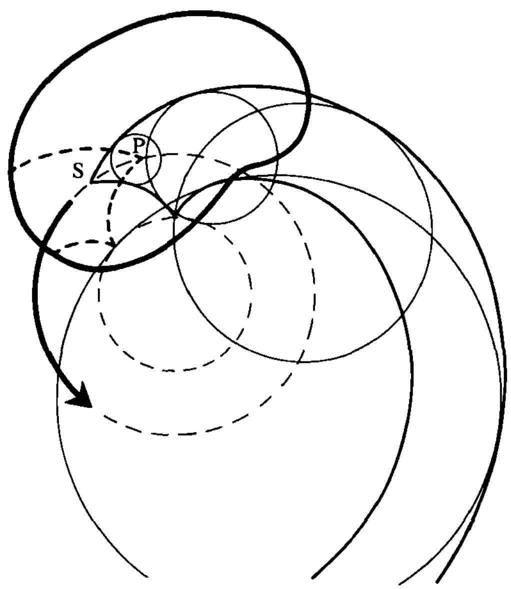

FIGURE 2. The counterpart of figure 1 for a rigidly rotating extended source. The larger of the two broken circles designates the orbit of the source point S and the smaller the sonic cylinder $r = c/\omega$. The envelope of the spherical wave fronts emanating from S and the bifurcation surface of the observation point P intersect the plane of the orbit along the curves that are shown, respectively, by the full and the broken lines.

source points outside this cone – whose own Mach cones do not enclose P – make no contribution towards the sound field at the observation point P, the source points inside it make two contributions, each at a different retarded time, towards the field at P. For source points *on* the inverted Mach cone the retarded times at which the two contributions are made coincide and the wavelets which arrive at P interfere constructively to form a singularity. Thus, in the space of the source points, the singularities of the Green's function – both for rectilinear and circular orbits – occur on an inverted caustic which, from the point of view of catastrophe theory, constitutes a bifurcation surface and so may be referred to as such.

In contrast to the rectilinear case, in which two wave fronts pass the observer inside the caustic and none outside it, the sound field in the circular case is non-zero also outside the caustic: because a source point in circular motion stays in a limited region of space for all time, it is eventually overtaken by the expanding wave fronts which have emanated from it at earlier times. If we consider the caustic generated by the point source (13) which appears in the equation for the Green's function and which has a non-zero density during only one revolution period, then, at any given observation time, three wave fronts propagating in different directions pass an observer located inside the caustic, and one wave front passes an observer outside it.† Each sheet of the caustic is formed by the constructive interference of only two of these waves; the third wave front always crosses the caustic at an angle and so represents that signal which is present on both sides of this surface. Correspondingly, the volume elements of the extended source shown in figure 2 which lie inside the bifurcation surface (depicted by the broken curves) influence the field at P at three different values of the retarded time and those outside it at one instant of an earlier time. The points *on* the bifurcation surface, for which the Doppler factor $|1-M_R|^{-1}$ diverges, are sources of the constructively interfering waves that not only arrive at P simultaneously but also are emitted at the same (retarded) time.

There is an infinitely sharp discontinuity in the value of the Green's function G_0 across the bifurcation surface. If we approach this surface from outside, the expression for G_0 in (9) has only one term and this term remains finite in the limit, but if we approach it from inside, then (9) has three terms and the two of these which are absent in the former expression diverge on the bifurcation surface. The conical apex of the bifurcation surface is an exceptional point: all three values of the Doppler factor $|1-M_R|^{-1}$ appearing in (9) remain finite and distinct as we approach this surface from opposite sides of $\hat{\varphi} = \hat{\varphi}_P$ along the circle $r = r_P$, $z = z_P$, but since R equals zero at this point, the discontinuity in G_0 is once again infinitely sharp.

Along the cusp curve, where the two sheets of the bifurcation surface meet, the rays associated with the field wavelets further focus and give rise to a higher-order singularity: at every point of this curve, one of the factors $1-M_R$ in the denominator of G_0 both vanishes and has a vanishing derivative with respect to φ, i.e. has a degenerate zero. This is a feature of the sound field from a circularly moving source which has no counterpart in that of a rectilinearly moving one.

Contrary to what would be expected from the analogy with the Mach cone, the present caustic lies entirely outside the dividing cylinder between the subsonic and supersonic trajectories (the sonic cylinder) no matter how close to this cylinder the source point may be (see figure 2). The closer the position of the source point to the sonic cylinder $r = c/\omega$, the smaller are the cross-sectional area of the caustic and the separation of the two segments of the U-shaped cusp curve. As the speed of the source

† While the Green's function G_0 is influenced by at most three values of the retarded time, the sound amplitude of a point source which executes many revolutions at a highly supersonic speed can be influenced by more than three waves that are simultaneously received at the observation point. As illustrated by figure 2 of Paper I, in the highly supersonic regime where the ordinates ϕ_+ and ϕ_- of the extrema of $g(\beta)$ are widely separated, the equation $g(\beta) = \phi$ has solutions also for values of ϕ which lie outside a single interval of length 2π.

point approaches the speed of sound, the volume enclosed by the caustic shrinks to zero and the whole surface collapses onto the cusp curve, whose two segments coalesce and lie in the plane of the orbit in this limit. This is, of course, what also happens to the bifurcation surface – which is the mirror image of the caustic – as the radial position, r_P, of the observation point approaches the sonic cylinder. The singularity of the Green's function is worst, that is the focusing of the rays of sound is sharpest, when the observation point is located within the source on a volume element whose speed (in the rotating frame) approaches the speed of sound from above. Not only do two of the factors in the denominator of G_0 have degenerate zeros at the observation point as this point is approached from $\hat{\varphi} < \hat{\varphi}_P$ along the circle $r = c/\omega$, $z = z_P$, but in fact the corresponding function M_R ceases to be analytic altogether when $r_P = c/\omega$ (see figure 6 of Paper I).

The sound amplitude due to a circularly moving point source has a stronger singularity than that due to a rectilinearly moving one because it is governed by an equation – (11) with the source term (13) – which is of the mixed type. In addition to possessing a cusp curve, on which two zeros of the factor $1-M_R$ coalesce, and an apex at which no derivatives of M_R exist, the present bifurcation surface collapses and so gives rise to further coalescence of the loci of various singularities of the Green's function when the observation point coincides with the sonic cylinder and r_P equals c/ω. That the bifurcation surface then disappears as r_P assumes a value smaller than c/ω, i.e. that the strong fields resulting from the constructive interference of the waves at the collapsed caustic are confined to the region outside the sonic cylinder and cannot propagate across this surface to reach an observer who is located just inside it, is a consequence of the fact that the sonic cylinder is the dividing surface between the domains of ellipticity and hyperbolicity of the relevant field equation. In fact, what in the four-dimensional (r, φ, z, t)-space appears as the bifurcation surface is from the point of view of the lower-dimensional $(r, \hat{\varphi}, z)$-space the ray conoid of (11) in its domain of hyperbolicity – which cannot, by definition, extend into the elliptic domain $r < c/\omega$ (Paper I, §6).

From this latter point of view, we can regard the variable $\hat{\varphi}$ as a time coordinate in $r > c/\omega$ and interpret (11) as the wave equation governing the generation and propagation of axisymmetric waves in a non-homogeneous medium for which the speed of sound varies like $(\omega^2 c^{-2} - r^{-2})^{-\frac{1}{2}}$ with the distance r from the axis of symmetry. In such a hypothetical medium, not only does the bending of the rays result in their convergence (on the hyperboloid generated by the cusp curves of the ray conoids mentioned above), but the unbounded increase in the local speed of sound would also cause the coalescence of the different parts of the envelope of the rays which originate from the vicinity of $r = c/\omega$ (onto the plane of symmetry of the hyperboloid) and lead to a higher-order focusing of the rays and hence also of the wave energy.

4. The speed-of-sound catastrophe

In the literature on rotor acoustics (see e.g. Farassat 1987 and the references therein) (4) is normally rewritten as

$$\rho(\boldsymbol{x}_P, t_P) = \frac{1}{4\pi c^2}\left[\frac{\partial^2}{\partial x_{Pi}\,\partial x_{Pj}}\int \mathrm{d}^3x\,\mathrm{d}t\,T_{ij}\,\theta(f)\,G - \frac{\partial}{\partial x_{Pi}}\int \mathrm{d}^3x\,\mathrm{d}t\,p_{ij}\frac{\partial f}{\partial x_j}\delta(f)\,G + \frac{\partial}{\partial t_P}\int \mathrm{d}^3x\,\mathrm{d}t\,\rho_0 v_i\frac{\partial f}{\partial x_i}\delta(f)\,G\right], \quad (17)$$

with $\boldsymbol{x}$ ranging over V, and t over $(-\infty, +\infty)$. The procedure which is used to obtain (17) from (4) involves the following steps: integration by parts to transfer the derivatives with respect to $\boldsymbol{x}$ and t from the source terms onto the Green's function, use of the fact that G is a function of $\boldsymbol{x}_{\mathrm{P}}-\boldsymbol{x}$ and $t_{\mathrm{P}}-t$ to equate these derivatives to the corresponding derivatives with respect to $\boldsymbol{x}_{\mathrm{P}}$ and t_{P} with a change in sign, and the interchanging of the orders of the differentiation with respect to $(\boldsymbol{x}_{\mathrm{P}}, t_{\mathrm{P}})$ and the integration over $(\boldsymbol{x}, t)$. From the point of view of the theory of classical functions, these mathematical manipulations are not permissible when the integral on the right-hand side of (4) has a singular integrand and does not converge uniformly. In particular, it is not permissible to transfer the derivatives from the source terms in (2) onto the Green's function in (4) in the supersonic regime, because the required integration by parts results in an integrated term which is singular and a new integrand which has a non-integrable singularity. Only Hadamard's finite parts (Courant & Hilbert 1962, p. 740) of the lengthy integrals thus obtained by Blackburn (1983), for instance, are relevant from the standpoint of the theory of generalized functions, and these finite parts are none other than the original integrals in which the derivatives operate on the source terms (see Whitham 1974, p. 221; Hoskins 1979).

So that any contributions there may be from the vicinities of the singularities of the integrand are not left out, in such cases one must exclude the singularities of the integrand from the domain of integration prior to these manipulations and proceed to the limit in which the volumes of the excluded regions shrink to zero only after having performed the integration. The singularity contributions to the value of the integral, if any, are of course the same whether obtained by this classical method or by means of the theory of generalized functions. To illustrate this point, let us consider Poisson's equation

$$\nabla^2 \rho = -4\pi s \tag{18}$$

and its familiar particular integral

$$\rho(\boldsymbol{x}_{\mathrm{P}}) = \int_V \frac{s(\boldsymbol{x})}{R}\,\mathrm{d}^3 x, \tag{19}$$

where the notation is the same as that in (3) and (4). If we were asked to verify that this integral is indeed a solution to (18), we would not be allowed – according to the classical theory – to take the Laplacian operator under the integral sign before excluding the singularity of the integrand at $R=0$ from V. Once we had done this, either by changing the limits of integration or equivalently by inserting the step function $\theta(R-\epsilon)$ with $0<\epsilon \ll 1$ in the integrand, we would also obtain a contribution from the boundary of the excluded region. In fact, as the following calculation shows, the value of $\nabla^2_{\mathrm{P}}\rho$ consists, in this case, solely of the contribution which arises from the vicinity of the singularity of the integrand:

$$\nabla^2_{\mathrm{P}}\rho = \lim_{\epsilon\to 0}\int \mathrm{d}^3 x\, s(\boldsymbol{x})\, \nabla^2_{\mathrm{P}}\left[\frac{\theta(R-\epsilon)}{R}\right] \tag{20}$$

$$= \lim_{\epsilon\to 0}\int \mathrm{d}^3 x\, s(\boldsymbol{x})\frac{\delta'(R-\epsilon)}{R} \tag{21}$$

$$= -4\pi s(\boldsymbol{x}_{\mathrm{P}}). \tag{22}$$

The same result follows from the less restrictive theory of generalized functions if this theory is applied properly: it is permissible to interchange the orders of the differentiation and the integration in the above calculation provided that $\nabla^2_{\mathrm{P}}(1/R)$ is regarded as a generalized function with the value $-4\pi\delta(\boldsymbol{x}-\boldsymbol{x}_{\mathrm{P}})$ rather than a classical function with the value zero.

While we do make use of the step function and the delta function for describing discontinuous functions and their derivatives, as we have done in (20) and (21), we shall nevertheless employ the classical method for differentiating the improper integrals that appear in the Ffowcs Williams–Hawkings equation. The calculation to be described in this section, which leads to the speed-of-sound catastrophe, is the counterpart of the following simpler calculation for a rectilinear motion. Figure 1 shows a localized extended source, i.e. a portion of a source distribution with the density $s(\boldsymbol{x},t)$ given in (2), which moves along one of the coordinate axes, e.g. the z-axis, supersonically: $s(\boldsymbol{x},t)$ depends on z and t in the combination $z-ut$ only, and $u > c$. In this rectilinear case, the sound amplitude at the observation point P does not receive any contributions from the source elements outside the inverted Mach cone (depicted by broken lines) that issues from P, and consists only of the superposition of the contributions of those volume elements of the source which lie within the bifurcation surface. The contribution from each source point *on* the bifurcation surface is infinite and so the integrand of the integral that represents the superposition is singular. But, as is well known (see e.g. Dowling & Ffowcs Williams 1983, p. 194), this singularity, which reflects the singularity in the density of a point source, is integrable; that is to say, the sound amplitude due to a finite-duration extended source is finite.

To calculate the gradient of the density perturbation that represents this sound amplitude, we can proceed in different ways. The simplest way, conceptually, is to integrate the product of the source density and the Green's function over the support of this product, i.e. over the intersection of the volume of the source and that of the inverted Mach cone issuing from P, and then directly to differentiate the resulting expression for the sound amplitude. Alternatively, we can choose the intersection of the source and a slightly displaced cone, that is contained within the original one, as the domain of integration so that we may interchange the orders of differentiation and integration, and after having integrated the product of the source density and the gradient of the Green's function and having calculated the contributions from the variable boundaries of the integration domain, retain only the part of the result that remains finite in the limit in which the two cones approach one another and coincide. A variant of this second method, which corresponds to Hadamard's method of taking finite parts, is to use integration by parts (permissible as long as the singularities of the integrand are excluded) to transfer the gradient from the Green's function onto the source density before performing the integration. The final answer is of course the same whichever method is used. However, the advantage of the methods in which the differentiation operator is taken under the integral sign – apart from the computational advantages – is that they enable us to single out any contributions which may arise from the vicinity of the singularity of the integrand.

In the simpler case of a rectilinearly moving extended source (figure 1), there are no contributions towards the value of the gradient of the sound amplitude from either the bifurcation surface in the supersonic regime, or from the point singularity at an interior observation point in the subsonic regime.

Let us now consider the corresponding problem in rotor acoustics. (The mathematical details of the results discussed in the remainder of this section appear in §4 of Paper I.) Figure 2 shows a similar localized extended source which rotates rigidly about the z-axis and for which the density $s(\boldsymbol{x},t)$, appearing in (2), is a function of φ and t in the combination $\varphi - \omega t$ only; the angular frequency of rotation, ω, is such that the source intersects the sonic cylinder $r = c/\omega$ (depicted by the smaller circle in this figure) and so moves in parts subsonically and in parts supersonically. Although the sound amplitude at P receives contributions from all source elements in this case, the bifurcation surface (shown in broken lines) again divides the volume of the source into two parts with differing influences on this amplitude; the source elements outside the bifurcation surface influence the sound amplitude measured at its conical apex at only a single instant of earlier time, while the source elements inside the surface influence this amplitude at three values of the retarded time. Amongst the source points *on* the bifurcation surface for which two of these retarded times are the same and the Doppler factor $|1-M_R|^{-1}$ is infinite, there are those on the cusp curve which approach the observation point at the speed of sound with zero acceleration, and so influence the sound amplitude at three coincident values of the retarded time. Even the singularities associated with these source points, however, are integrable for all values of r_P including $r_P = c/\omega$ and, as in the rectilinear case, the sound amplitude from an extended source is itself finite.

The two components $\partial\rho/\partial z_P$ and $\partial\rho/\partial\hat{\varphi}_P$ of the gradient of the sound amplitude, too, are like ρ everywhere finite. This is implied by the fact that the equations governing these two components, obtained by differentiating (11), have the same structure as the one governing ρ itself. To infer this result directly from (7), Hadamard's finite parts of the differentiated integrals must be adopted: we must begin by enclosing various parts of the bifurcation surface inside shells whose volumes are excluded from $\hat{V}$, then take the derivatives under the integral sign and use integration by parts and the fact that G_0 is a function of $z - z_P$ and $\hat{\varphi} - \hat{\varphi}_P$ to transfer the derivatives onto the source terms, and finally note that when the volumes of the excluded shells shrink to zero, the resulting expressions for $\partial\rho/\partial z_P$ and $\partial\rho/\partial\hat{\varphi}_P$ differ from the expression for ρ only in that s in them is replaced by $\partial s/\partial z$ or $\partial s/\partial\hat{\varphi}$.

The component $\partial\rho/\partial r_P$ of he gradient of the sound amplitude, on the other hand, behaves differently. In the case of wave equation (1), it is always possible to work with Cartesian coordinates so that the structure of the d'Alembertian operator is not changed by differentiation, but in rotor acoustics the change that occurs when the operator on the left-hand side of (11) is differentiated with respect to r cannot be avoided by a different choice of coordinates, and is an indication of the fact that $\partial\rho/\partial r_P$ is not related to the source term $\partial s/\partial r$ via the familiar Green's function G_0. When we insert a step function H in the integrand of (7) to exclude the bifurcation surface from the domain of integration and then differentiate this equation with respect to r_P by the method outlined above, we find that, because the variable r_P enters G_0 not only in the combination $r - r_P$ but also on its own, the operator $\partial/\partial r_P$ which acts on $G_0 H$ cannot be replaced with $-\partial/\partial r$ and so wholly transferred onto the source term. Once we have integrated the terms which are common to $\partial(G_0 H)/\partial r_P$ and $-\partial(G_0 H)/\partial r$ by parts, we are left with the remaining terms of $\partial(G_0 H)/\partial r_P$ which, in the limit where $H = 1$, play the role of a new Green's function.

To decide whether the higher-order singularity of this new Green's function which appears in the expression for $\partial\rho/\partial r_P$ is integrable, let us for the moment set the source density s equal to a constant. The integration over $\hat{\varphi}$ can then be performed explicitly and we find that certain terms in the integrated expression which arise from the boundaries of the excluded region are still singular after this first integration: in the limit where the volume of the excluded region reduces to zero, the contribution of the source points which approach the cusp curve from inside the bifurcation surface diverges. However, when the remaining integrations are performed over a small area containing the apex $r = r_{\mathrm{P}}$, $z = z_{\mathrm{P}}$, of the projection of the bifurcation surface onto the (r, z)-plane, the end result for the value of the volume integral that represents $\partial\rho/\partial r_{\mathrm{P}}$ turns out to be finite and is proportional to $(r_{\mathrm{P}} - c/\omega)^{-\frac{1}{4}}$ for $0 < (r_{\mathrm{P}}\,\omega/c) - 1 \ll 1$. Because the first term in (2) for s, i.e. the quadrupole term which represents the density of a volume distribution, does in act assume a constant value in a sufficiently small neighbourhood of the observation point $r = r_{\mathrm{P}}$, $z = z_{\mathrm{P}}$, $\hat{\varphi} = \hat{\varphi}_{\mathrm{P}}$, this result implies that the gradient of the sound amplitude which arises from a quadrupole source is finite everywhere except at the sonic cylinder where its radial component, $\partial\rho/\partial r_{\mathrm{P}}$, diverges like $(r_{\mathrm{P}} - c/\omega)^{-\frac{1}{4}}$ as $r_{\mathrm{P}}\,\omega/c \to 1+$.

The above catastrophe, i.e. the sudden and infinitely large change in the value of $\partial\rho/\partial r_{\mathrm{P}}$ that occurs as the control parameter $r_{\mathrm{P}}\,\omega/c$ crosses the critical value 1, arises from the contribution of those source points which lie on the cusp curve of the bifurcation surface issuing from the observation point, in the limit where the two segments of this U-shaped curve approach one another and coalesce. Since the curve resulting from the collapse of the bifurcation surface lies in the plane $z = z_{\mathrm{P}}$, the quantity $\partial\rho/\partial r_{\mathrm{P}}$ diverges even more strongly when the source is distributed over the equatorial plane and the observation point coincides with the intersection of this plane and the sonic cylinder. If the loading and the thickness noise terms, i.e. the second and the third terms in (2), which are given by integrals over the blade surface $f(r, z, \hat{\varphi}) = 0$, are replaced, as is customary (cf. Farassat 1986), with integrals over a plane representing the mean surface of the rotor blade, then the value of $\partial\rho/\partial r_{\mathrm{P}}$ on this plane diverges like $(r_{\mathrm{P}} - c/\omega)^{-\frac{3}{4}}$ as $r_{\mathrm{P}}\,\omega/c \to 1+$ (see Appendix A).

In this section, our analysis has been based on the particular form of the Ffowcs Williams–Hawkings equation given in (7). But as was pointed out in §2, the singularity under discussion also follows from the Duhamel's form of the retarded potential, (16). The corresponding calculation, outlined in Appendix B, shows that the divergent contributions to the value of $\partial\rho/\partial r_{\mathrm{P}}$, though ostensibly arising from the integrand in the case of (7) and from the limits of integration in the case of (16), are in both cases caused by the discontinuous variation of the Green's function G_0, or equivalently the Jacobian $\mathrm{d}\varphi/\mathrm{d}\hat{\varphi}$, across the bifurcation surface, and that they have precisely the same asymptotic value.

5. The equivalent Cauchy problem for the homogeneous wave equation

Just as the Cauchy data for the solution of the homogeneous wave equation can be replaced by suitably chosen impulsive source terms (Morse & Feshbach 1953, p. 837), so the solution of the inhomogeneous equation (1) discussed here can be expressed as that of an equivalent Cauchy problem for the homogeneous wave equation. To formulate this equivalent problem, we may begin by writing the source term s of (1) as a superposition of impulses:

$$s(r, z, \hat{\varphi}) = \int_{-\infty}^{+\infty} \mathrm{d}\hat{\varphi}_0\, s(r, z, \hat{\varphi}_0)\, \delta(\hat{\varphi} - \hat{\varphi}_0), \tag{23}$$

i.e. by invoking Duhamel's principle (see Courant & Hilbert 1962, p. 552). By virtue of the linearity of the wave equation, it is then sufficient to discuss the Cauchy problem for a source term of the form $s(r, z, \hat{\varphi}_0)\, \delta(\hat{\varphi} - \hat{\varphi}_0)$ only. (For a discussion of the inhomogeneous problem for such two-dimensional rotating sources that lie in a meridional plane, see Appendix C.)

Since the Cauchy data required to duplicate such a source have to be prescribed on a curved hypersurface in the four-dimensional space–time, it is for the purposes of the present section more convenient to introduce the Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

and rewrite the equations for the wave amplitude (equation (1)) and the Green's function G (equation (3)) in the forms

$$\eta^{\mu\nu} \frac{\partial^2 \rho}{\partial x^\mu \, \partial x^\nu} = -\frac{1}{c^2} s \tag{25}$$

and
$$\eta^{\mu\nu} \frac{\partial^2 G}{\partial x^\mu \, \partial x^\nu} = -4\pi c \delta(x^\mu - x^\mu_{\mathrm{P}}), \tag{26}$$

where $x^\mu = (ct, x, y, z)$ are the space–time coordinates, the Greek indices μ, ν, etc. take on the values 0, 1, 2 and 3, and repeated indices are to be summed over. If we now multiply (25) and (26) by G and ρ, respectively, subtract the two equations and integrate the resulting expression over the space–time domain D with the boundary ∂D, we obtain the following generalization of equation (7.3.5) of Morse & Feshbach (1953):

$$\rho(x^\mu_{\mathrm{P}}) = \frac{1}{4\pi c^3} \int_D s\, G \, \mathrm{d}^4 x + \frac{1}{4\pi c} \oint_{\partial D} \left(G \frac{\partial \rho}{\partial x^\nu} - \rho \frac{\partial G}{\partial x^\nu} \right) \mathrm{d}S^\nu. \tag{27}$$

the 4-vector $\mathrm{d}S^\nu$ in this equation represents the volume element on the (generally curved) 3-surface ∂D.

The solution to the inhomogeneous wave equation for a source density of the form $s(r, z, \hat{\varphi}_0)\, \delta(\hat{\varphi} - \hat{\varphi}_0)$ in unbounded space–time is given by the first term in (27):

$$\rho(x^\mu_{\mathrm{P}}) = \frac{1}{4\pi c^2 \omega} \int_{\hat{\varphi} = \hat{\varphi}_0} s(r, z, \hat{\varphi}_0)\, G\, r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z. \tag{28}$$

Here, the integration extends over the hypersurface $\hat{\varphi} = \hat{\varphi}_0$ whose parametric equation in terms of the interior coordinates (r, φ, z) can be written as

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \quad t = (\varphi - \hat{\varphi}_0)/\omega. \tag{29}$$

Since in this case ∂D lies at infinity, the second term in (27) makes no contribution. The solutions to the homogeneous wave equation, on the other hand, consist entirely of the boundary terms in (27). So, to obtain the same result (equation (28)) by solving the homogeneous wave equation, i.e. from the second term in (27), we must prescribe the Cauchy data on $\hat{\varphi} = \hat{\varphi}_0$ and close the boundary ∂D by means of a semi-spherical hypersurface of infinite radius in $\hat{\varphi} < 0$ (which corresponds to $t > 0$).

The volume element on the 3-surface $\hat{\varphi} = \hat{\varphi}_0$ can be calculated from (29) as follows:

$$\mathrm{d}S_\mu = \epsilon_{\mu\nu\sigma\lambda} \frac{\partial x^\nu}{\partial r} \frac{\partial x^\sigma}{\partial \varphi} \frac{\partial x^\lambda}{\partial z} \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z = n_\mu \, r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z, \tag{30}$$

where
$$n_\mu = \left(1, \frac{c}{r\omega}\sin\varphi, -\frac{c}{r\omega}\cos\varphi, 0\right), \tag{31}$$

which is proportional to $\partial\hat{\varphi}/\partial x^\mu$, represents the normal to the hypersurface, and $\epsilon_{\mu\nu\sigma\lambda}$ is the Levi–Civita tensor. Substituting (30)in the second term of (27) and comparing the resulting integrand with that in (28), we can now see that the required Cauchy data are

$$\rho|_{\hat{\varphi}=\hat{\varphi}_0} = 0, \tag{32}$$

and

$$n^\nu \frac{\partial\rho}{\partial x^\nu}\bigg|_{\hat{\varphi}=\hat{\varphi}_0} = -\frac{1}{c\omega} s(r,z,\hat{\varphi}_0). \tag{33}$$

(The outward normal to ∂D is $-n^\nu$). On $\hat{\varphi}=\hat{\varphi}_0$ we have $\mathrm{d}\varphi = \omega\,\mathrm{d}t$ and so (32), in conjunction with the fact that $\partial\rho/\partial r$ and $\partial\rho/\partial z$ are derivatives along directions interior to $\hat{\varphi}=\hat{\varphi}_0$, implies that

$$\frac{\partial\rho}{\partial r}\bigg|_{\hat{\varphi}=\hat{\varphi}_0} = 0, \quad \frac{\partial\rho}{\partial z}\bigg|_{\hat{\varphi}=\hat{\varphi}_0} = 0, \quad \left(\frac{1}{\omega}\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial\varphi}\right)_{\hat{\varphi}=\hat{\varphi}_0} = 0. \tag{34}$$

Equation (33), therefore, assumes the form

$$\frac{\partial\rho}{\partial t}\bigg|_{\hat{\varphi}=\hat{\varphi}_0} = \frac{\omega s(r,z,\hat{\varphi}_0)}{\omega^2-c^2/r^2}, \tag{35}$$

once (31) is used to express $n^\mu = \eta^{\mu\nu} n_\nu$ explicitly.

The reason for the divergence of the right-hand side of (35) at $r = c/\omega$ is that, as indicated by

$$n_\mu n^\mu = c^2/(r\omega)^2 - 1, \tag{36}$$

the 4-vector n^μ, which is time-like in $r > c/\omega$, undergoes a change in type across the sonic cylinder and becomes space-like in $r < c/\omega$. In other words, the vector normal to the 3-surface $\hat{\varphi}=\hat{\varphi}_0$ becomes null, and so coincident with the normal to a characteristic hypersurface, at all points on the 2-surface $r = c/\omega$.

To solve the homogeneous wave equation with the above Cauchy data, it is more convenient to adopt the independent variables $(r,\varphi,z,\hat{\varphi})$ so that the data are given on a coordinate surface. In these coordinates, the wave equation and the data (32)–(35) have the forms

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho}{\partial r}\right)+\frac{\partial^2\rho}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2\rho}{\partial\varphi^2}+\frac{2}{r^2}\frac{\partial^2\rho}{\partial\varphi\,\partial\hat{\varphi}}+\left(\frac{1}{r^2}-\frac{\omega^2}{c^2}\right)\frac{\partial^2\rho}{\partial\hat{\varphi}^2} = 0, \tag{37}$$

$$\rho|_{\hat{\varphi}=\hat{\varphi}_0} = 0, \quad \frac{\partial\rho}{\partial\hat{\varphi}}\bigg|_{\hat{\varphi}=\hat{\varphi}_0} = -\frac{s(r,z,\hat{\varphi}_0)}{\omega^2-c^2/r^2}, \tag{38}$$

and the derivatives with respect to r, z and φ are all along directions interior to the hypersurface $\hat{\varphi}=\hat{\varphi}_0$. The exterior derivatives $\partial^2\rho/\partial\hat{\varphi}^2|_{\hat{\varphi}=\hat{\varphi}_0}$, $\partial^3\rho/\partial\hat{\varphi}^3|_{\hat{\varphi}=\hat{\varphi}_0}$, etc., which are needed for evolving the solution away from this hypersurface, can now be determined by evaluating (37) and its successive derivatives at $\hat{\varphi}=\hat{\varphi}_0$. Because $\partial\rho/\partial\hat{\varphi}|_{\hat{\varphi}=\hat{\varphi}_0}$ is independent of φ, the insertion of (38) in (37) yields $\partial^2\rho/\partial\hat{\varphi}^2|_{\hat{\varphi}=\hat{\varphi}_0} = 0$. When this and the initial data are inserted in the derivative of (37) with respect to $\hat{\varphi}$, however, we obtain a value for the next coefficient in the Taylor expansion of ρ,

$$\frac{\partial^3\rho}{\partial\hat{\varphi}^3}\bigg|_{\hat{\varphi}=\hat{\varphi}_0} = \left(\frac{1}{r^2}-\frac{\omega^2}{c^2}\right)^{-1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}\right]\frac{s(r,z,\hat{\varphi}_0)}{\omega^2-c^2/r^2}, \tag{39}$$

which diverges at $r = c/\omega$ for an arbitrary source density $s(r, z, \hat{\varphi}_0)$. In fact the remaining derivatives, $\partial^m \rho/\partial\hat{\varphi}^m|_{\hat{\varphi}=\hat{\varphi}_0}$ for $m > 3$, are all divergent at $r = c/\omega$ and the higher the value of m, the higher is the order of their singularities.

The coefficients in the Taylor expansion of the solution to the above Cauchy problem diverge at the sonic cylinder simply because the Cauchy data are prescribed on a hypersurface which is locally characteristic at all points of the 2-surface $r = c/\omega$: according to (36), the 3-surface $\hat{\varphi} = \hat{\varphi}_0$ is space-like in $r > c/\omega$, is null, i.e. characteristic, at $r = c/\omega$, and is time-like in $r < c/\omega$. It is well known that unless the data are also characteristic at points where the initial hypersurface becomes characteristic, the Cauchy problem cannot have a solution whose derivatives are regular; for an illustrative example of this type of singularity, see the discussion in Chapter 1 of Bleistein (1984). In the present case, the constraint implied by (39) and the corresponding expressions for the higher-order derivatives $\partial^m \rho/\partial\hat{\varphi}^m|_{\hat{\varphi}=\hat{\varphi}_0}$ is that unless the function $s(r, z, \hat{\varphi}_0)$ appearing in the Cauchy data (38) approaches zero faster than all powers of $r - c/\omega$, some of the coefficients in the Taylor expansion of ρ diverge at $r = c/\omega$. Indeed, the singularity of the original inhomogeneous problem discussed in the preceding section is removed if the source density $s(r, z, \hat{\varphi})$ vanishes exponentially at the sonic cylinder (cf. Paper I, §5). However, because the data are still prescribed on a hypersurface which is in parts time-like, the Cauchy problem in question remains ill-posed even after such a modification (see Bleistein 1984, p. 148; and Morse & Feshbach 1953, p. 683).

Although appearing in somewhat different guises, the singularities of the original inhomogeneous problem discussed in §4 and those of the equivalent homogeneous problem discussed here have exactly the same origins. The fact that the initial hypersurface associated with the homogeneous problem becomes characteristic is intimately related to the fact that the solution to both problems possesses the following symmetry earlier expressed in (12):

$$u^\mu \frac{\partial \rho}{\partial x^\mu} = 0, \tag{40}$$

where
$$u^\mu = \left(1, -\frac{r\omega}{c}\sin\varphi, \frac{r\omega}{c}\cos\varphi, 0\right). \tag{41}$$

As can be seen from the invariant quantity

$$u_\mu u^\mu = -1 + r^2\omega^2/c^2, \tag{42}$$

the 4-vector u^μ, like the normal to the initial hypersurface n^μ, is time-like in $r > c/\omega$ and space-like in $r < c/\omega$. That the symmetry expressed by the directional derivative (40) is with respect to time in $r > c/\omega$ and with respect to space in $r < c/\omega$, on the other hand, is the reason why the wave equation subject to this symmetry, i.e. (11), is hyperbolic in $r > c/\omega$ and elliptic in $r < c/\omega$ (see also §6 of Paper I). It is thus immaterial whether we regard the singularity as arising from the ill-posed character of the initial data or from the mixed nature of the governing field equation; these are two aspects of a single feature.

The singularity in question does in fact occur on the characteristics of the wave equation as expected. We already know from the solution to the inhomogeneous problem that, on the one hand, this singularity stems from the singularity of the Green's function G_0, and that, on the other hand, the bifurcation surface on which G_0 is singular is a solution of the eikonal equation and so a characteristic manifold of the wave equation (Paper I, §6). Since the initial data (38) lead to a solution which possesses the symmetry (12), the characteristics responsible for the propagation of the singularity are only those which are stationary in the rotating frame. We shall see below that at any given time the points on the sonic cylinder each belong to a specific member of the set of characteristic 2-surfaces whose motion is entirely azimuthal. Such characteristics only exist in $r > c/\omega$, so that the sonic cylinder in fact constitutes the locus of the edges of regression, i.e. the envelope, of the set of azimuthally propagating wave fronts.

The characteristics of the wave equation that remain stationary in the rotating frame are those hypersurfaces, $\psi(ct, x, y, z) = \text{const.}$, which in addition to having null normals,

$$\eta^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu} = 0, \tag{43}$$

possess the symmetry expressed in (12) and (40). The eikonal equation (43) for a function ψ that satisfies condition (12) has the following form in cylindrical polar coordinates:

$$\left(\frac{\partial\psi}{\partial\hat{r}}\right)^2 + \left(\frac{\partial\psi}{\partial\hat{z}}\right)^2 + \left(\frac{1}{\hat{r}^2} - 1\right)\left(\frac{\partial\psi}{\partial\hat{\varphi}}\right)^2 = 0, \tag{44}$$

where $\hat{r} \equiv r\omega/c$ and $\hat{z} \equiv z\omega/c$. The integration of this differential equation is considerably simplified by a Legendre transformation (Courant & Hilbert 1962). Its general solution, as can be verified by direct substitution, is given by

$$\psi(\hat{r}, \hat{z}, \hat{\varphi}) = -f(\xi_2, \xi_3) + \xi_2\frac{\partial f}{\partial\xi_2} + \xi_3\frac{\partial f}{\partial\xi_3}, \tag{45}$$

$$\hat{r} = \frac{\pm\xi_3}{(\xi_3^2 - \xi_2^2 - \xi_1^2)^{\frac{1}{2}}}, \tag{46}$$

$$\hat{z} = \frac{\pm\xi_1\xi_2\xi_3}{(\xi_3^2 - \xi_2^2)(\xi_3^2 - \xi_2^2 - \xi_1^2)^{\frac{1}{2}}} + \frac{\partial f}{\partial\xi_2}, \tag{47}$$

$$\hat{\varphi} = \pm\arcsin\frac{\xi_1}{(\xi_3^2 - \xi_2^2)^{\frac{1}{2}}} \mp \frac{\xi_1\xi_3^2}{(\xi_3^2 - \xi_2^2)(\xi_3^2 - \xi_2^2 - \xi_1^2)^{\frac{1}{2}}} + \frac{\partial f}{\partial\xi_3}, \tag{48}$$

in which $f(\xi_2, \xi_3)$ is an arbitrary function of the two variables ξ_2 and ξ_3. The intermediary variables (ξ_1, ξ_2, ξ_3) are in fact the components of the vector normal to the characteristic surface: $\xi_1 = \partial\psi/\partial\hat{r}$, $\xi_2 = \partial\psi/\partial\hat{z}$ and $\xi_3 = \partial\psi/\partial\hat{\varphi}$.

The fact that all terms in (44) are positive when $\hat{r} < 1$ already implies that the above solutions exist only outside the sonic cylinder. These azimuthally propagating characteristics, which are stationary 3-surfaces in the 4-dimensional $(\psi, \hat{r}, \hat{z}, \hat{\varphi})$-space, possess an envelope consisting of the 2-surface $\hat{r} = 1$, since the Jacobian

$$\begin{aligned}\frac{\partial(\hat{r}, \hat{z}, \hat{\varphi})}{\partial(\xi_1, \xi_2, \xi_3)} = {}& \frac{\xi_1\xi_3}{(\xi_3^2 - \xi_2^2 - \xi_1^2)^{\frac{3}{2}}}\left[\frac{\partial^2 f}{\partial\xi_2^2}\frac{\partial^2 f}{\partial\xi_3^2} - \left(\frac{\partial^2 f}{\partial\xi_2\,\partial\xi_3}\right)^2\right] \\ & + \frac{\xi_1^2(\xi_3^2 + \xi_2^2) - \xi_2^2(\xi_3^2 - \xi_2^2)}{(\xi_3^2 - \xi_2^2)^2(\xi_3^2 - \xi_2^2 - \xi_1^2)^2}\left(\xi_3^2\frac{\partial^2 f}{\partial\xi_3^2} + 2\xi_2\xi_3\frac{\partial^2 f}{\partial\xi_2\,\partial\xi_3} + \xi_2^2\frac{\partial^2 f}{\partial\xi_2^2}\right)\end{aligned} \tag{49}$$

vanishes at $\hat{r} = 1$ where, according to (46), we have $\xi_1 = \xi_2 = 0$. For any given $f(\xi_2, \xi_3)$, the associated characteristic has an edge of regression along the curve $\hat{r} = 1$, $\hat{z} = \partial f/\partial\xi_2|_{\xi_2=0}$, $\hat{\varphi} = \partial f/\partial\xi_3|_{\xi_2=0}$, $\psi = (-f + \xi_3\,\partial f/\partial\xi_3)_{\xi_2=0}$. The projection of this curve onto the subspace $\psi = \text{const.}$ is the point at which the surface $\psi(r, z, \hat{\varphi}) = \text{const.}$ touches the sonic cylinder. Since both $\partial\psi/\partial\hat{r}$ and $\partial\psi/\partial\hat{z}$ vanish at this point, the normal to the 2-surface $\psi(r, z, \hat{\varphi}_0) = \text{const.}$ is, as illustrated by the following particular solution

$$\psi = \hat{\varphi} - (\hat{r}^2 - 1)^{\frac{1}{2}} + \arcsin(1 - \hat{r}^{-2})^{\frac{1}{2}} = 0, \tag{50}$$

purely azimuthal at the sonic cylinder. That is to say, each characteristic $\psi = \text{const.}$ has a cusp at the point (or points) where it meets the sonic cylinder.

The caustic shown in figure 2, which constitutes the locus of the singularities in the sound amplitude of a point source, is a specific member of the set of azimuthally propagating characteristics described here and is given by (45)–(48) for a particular $f(\xi_1, \xi_2)$ (see Paper I, §6). The sonic cylinder, on which the gradient of the sound amplitude from an extended source and the solution to the present Cauchy problem are singular, is not itself a characteristic, but constitutes the envelope of all azimuthally propagating characteristics. The occurrence of a singularity on the envelope or the edge of regression of characteristics, across which an equation can change from hyperbolic to elliptic is in fact frequently encountered in solutions to ill-posed problems for equations of the mixed type (see Bitsadze 1964).

6. The nonlinear regime of the theory

Within the framework of the linearized theory, the source term s is known up to the first order in the perturbation quantities and the expression given in (7) for ρ represents the solution to (11). Since the radial component $\partial\rho/\partial r_{\mathrm{P}}$ of the sound amplitude is a measurable quantity that cannot assume an infinite value, its singularity is a physically unacceptable prediction of (11) which must be interpreted as signifying the breakdown of the linearized theory. Clearly, here is yet another example of a transonic flow in which the linearized theory breaks down even under circumstances where the perturbations are small (cf. Moulden 1984).

In the nonlinear regime, where the source term s is not known, (7) is an integral representation of the differential equation (11). However, the insertion of (7) into (11) would not result in an identity unless all second derivatives of ρ exist. The infinite discontinuity in the value of $\partial\rho/\partial r_{\mathrm{P}}$ at the sonic cylinder, which occurs for any quasi-steady source term of the form $s = s(r, z, \hat{\varphi})$ whether known or not, implies that within the transonic domain of the flow, (7) and (11) are not in fact equivalent. Any description of the flow in this regime must be based on some version of the original differential equation.

Rather than going back to (1), let us follow the common practice in the literature (cf. Caradonna & Isom 1976) and assume that the flow is irrotational, so that we can describe it by means of a single partial differential equation of the second order. In the case of a potential flow, where $\boldsymbol{u} = \nabla\phi$, Bernoulli's equation and the equation of continuity jointly yield

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi + \frac{\partial}{\partial t}|\nabla\phi|^2 + \nabla\phi\cdot\nabla(\tfrac{1}{2}|\nabla\phi|^2) = 0. \tag{51}$$

To express the speed of sound in this equation as a function of the velocity potential ϕ, we need to introduce an equation of state for the fluid; with an adiabatic equation of state, Bernoulli's equation becomes

$$c^2 = c_\infty^2 - (\gamma - 1)\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2\right), \tag{52}$$

where γ is the ratio of the specific heats at constant pressure and constant volume, and c_∞ is the speed of sound in the undisturbed medium. Equations (51) and (52), for a quasi-steady potential that satisfies the symmetry condition (12), have the forms

$$(u_r^2-c^2)\frac{\partial^2\phi}{\partial r^2}+2u_r\left(\frac{u_\varphi}{r}-\omega\right)\frac{\partial^2\phi}{\partial r\,\partial\hat{\varphi}}+2u_r u_z\frac{\partial^2\phi}{\partial r\,\partial z}+\left[\left(\frac{u_\varphi}{r}-\omega\right)^2-\frac{c^2}{r^2}\right]\frac{\partial^2\phi}{\partial\hat{\varphi}^2}$$
$$+2u_z\left(\frac{u_\varphi}{r}-\omega\right)\frac{\partial^2\phi}{\partial\hat{\varphi}\,\partial z}+(u_z^2-c^2)\frac{\partial^2\phi}{\partial z^2}=\frac{u_r}{r}(u_\varphi^2+c^2), \tag{53}$$

and

$$c^2=c_\infty^2+(\gamma+1)\,(r\omega u_\varphi-\tfrac{1}{2}u^2), \tag{54}$$

where $u_r=\partial\phi/\partial r$, $u_\varphi=r^{-1}\,\partial\phi/\partial\hat{\varphi}$ and $u_z=\partial\phi/\partial z$ are the cylindrical components of $\boldsymbol{u}$, and $u\equiv|\boldsymbol{u}|$. As we would expect, in the linear regime $u\ll c$, the second-order operator in (53) reduces to that in (11).

The above quasi-linear partial differential equation for ϕ is elliptic or hyperbolic according to whether the following determinant – whose elements consist of the coefficients of the second-order derivatives in this equation – is positive or negative (see Courant & Hilbert 1962, p. 181):

$$\begin{vmatrix} u_r^2-c^2 & u_r\left(\dfrac{u_\varphi}{r}-\omega\right) & u_r u_z \\ u_r\left(\dfrac{u_\varphi}{r}-\omega\right) & \left(\dfrac{u_\varphi}{r}-\omega\right)^2-\dfrac{c^2}{r^2} & u_z\left(\dfrac{u_\varphi}{r}-\omega\right) \\ u_r u_z & u_z\left(\dfrac{u_\varphi}{r}-\omega\right) & u_z^2-c^2 \end{vmatrix}=-\frac{c^4}{r^2}[(u_\varphi-r\omega)^2+u_r^2+u_z^2-c^2], \tag{55}$$

i.e. whether the fluid velocity relative to the blade-fixed coordinates is subsonic or supersonic. On the surface of parabolic degeneracy, where (53) undergoes a change in type, the right-hand side of (55) is zero and so

$$\frac{c^2}{c_\infty^2}=\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\frac{r^2\omega^2}{c_\infty^2}\right) \tag{56}$$

(see (54) and (55)). This surface coincides with the sonic cylinder in the linear regime but, in general, has an irregular shape which is determined by the local values of u and c rather than by c_∞ alone. In fact, an important nonlinear effect is the departure of the speed of sound c from its undisturbed value c_∞ and, as (56) shows, we have $c<c_\infty$ at any points of the surface of parabolic degeneracy which lie inside the sonic cylinder $r=c_\infty/\omega$.

This provides an explanation for the experimental results of Schmitz & Yu (1981) according to which there is a breakdown in the linearized theory before the tip Mach number $r\omega/c_\infty$ of a helicopter blade attains the value 1. The main effect of the rotational motion of a thin hovering blade on the flow around it is to impose the quasi-static constraint (12). However, no quasi-static flows exist for which the amplitudes of the perturbations in the flow variables remain small at the sonic cylinder. On the one hand, the imposition of the quasi-static constraint in the vicinity of the sonic cylinder is accompanied by the generation of large-amplitude perturbations, and on the other hand the position of the surface of parabolic degeneracy is changed in the presence of such perturbations. As a result, the fluid velocity relative to the blade-fixed coordinates (whose zeroth-order value is $r\omega$) can equal the local speed of sound $c(< c_\infty)$ at points where $r < c_\infty/\omega$. Thus the nonlinear effects come into play when the tip of the blade pierces the surface of parabolic degeneracy rather than when it pierces the sonic cylinder (as suggested by Schmitz & Yu 1986).

Numerical calculations based on (53) confirm that, as suggested by the infinitely large gradient of the sound amplitude predicted by the linearized theory, the flow in the transonic region includes shock discontinuities (Caradonna & Isom 1976). Far from the transonic region, nonlinear effects are unimportant and the flow is once again governed by (11). One can use the near-zone flow field found from the numerical solution of (53) to specify the source term s in (11) and subsequently solve this equation in the radiation zone (see Hanson & Fink 1979). However, a more appropriate solution to (11) is in this case provided by a Kirchhoff integral over a surface close to the inner boundary of the domain of hyperbolicity of the flow (cf. Isom *et al.* 1987; Schulten 1988). One can use the computed flow field in the transonic region to specify the initial data on a surface just outside the surface of parabolic degeneracy, and subsequently solve the Cauchy boundary value problem for (11). (Of course, the Kirchhoff integral solves a Cauchy problem for which the data are set on a time-like surface, and as a result, the dependence of the solution on the initial data might not be continuous.)

The Kirchhoff integral formulation is more appropriate because the acoustic field is in the present case influenced by the flow in the near zone only to the same extent that the conditions at the base of the hyperbolic domain are influenced. From the point of view of an observer in the lower dimensional $(r,\hat{\varphi},z)$-space of (11), who cannot detect waves that do not depend on φ and t in the combination $\varphi-\omega t$, the only signals reaching infinity are those which originate at points outside the surface of parabolic degeneracy (see §6 of Paper I). The domain of dependence of the entire radiation field, therefore, consists of the outer boundary of the transonic region. The shock fronts which cross this boundary correspond to discontinuities in the Cauchy initial data. It is well known, on the other hand, that the discontinuities in the data propagate along the characteristic surfaces of the governing hyperbolic equation. The bicharacteristic curves or rays associated with (11) (given in equation (90) of Paper I) that originate at points where the initial data is discontinuous form a characteristic surface extending to infinity on which the flow itself is discontinuous. The characteristic surfaces of (11) constituting the loci of the flow discontinuities are stationary in the $(r,\hat{\varphi},z)$-space, but they rotate around the z-axis with the angular velocity ω in the (r,φ,z,t)-space (see equations (45)–(48)).

The experimentally observed phenomenon which Schmitz & Yu (1986) call the delocalization of shock discontinuities is in fact the effect described above: the near-field shocks that are formed concurrently with the breakdown of the linearized theory in the transonic region act as discontinuities in the Cauchy data on a surface close to the inner boundary of the domain of hyperbolicity of the flow and so propagate along the characteristic surfaces of the governing field equation and appear as acoustic shocks in the far-field radiation zone. Thus the sources of the detected impulsive noise are the transonic shock discontinuities near the sonic cylinder; acoustic waveforms develop discontinuities only when the tip Mach number of the rotor tends sufficiently close to unity for these near-field shocks to form and to intersect the surface of parabolic degeneracy.

From the standpoint of the Ffowcs Williams–Hawkings equation, these sources belong to the quadrupole term in (2). Loci of the discontinuities of the monopole and the dipole terms, which consist of the intersections of the near-field shock fronts with the blade surface, are curves which in general cross the surface of parabolic degeneracy at isolated points. Such pointwise discontinuities in the Cauchy initial data propagate into the hyperbolic region, and so into the far zone, along isolated rays. On the other hand, the discontinuity in the quadrupole term which consists of an entire shock front crosses the domain of dependence of the hyperbolic region (e.g. a surface just outside the surface of parabolic degeneracy) along a curve and so propagates into the supersonic region along a characteristic surface. It is for this reason that theory and experiment are brought into agreement once the discontinuous quadrupole sources are included in the calculation of the acoustic radiation field (Hanson & Fink 1979; Schmitz & Yu 1986).

7. Concluding remarks

Mathematically, the results discussed in §§2–6 derive from the assumption central to rotor acoustics that the flow in the blade-fixed coordinate frame is steady and hence the equation governing this flow is of the mixed type. The agreement between theory and experiment substantiates this assumption and confirms that the long-term rotational motion of a hovering blade does in fact produce a flow which when viewed in the blade-fixed frame is essentially time-independent. But if it were possible to decouple the motion of the blade from the symmetry of the flow, i.e. to prevent the rotational motion of the blade from imposing the quasi-static condition (12) on the surrounding flow, then there would be neither a transonic region with shock discontinuities in the near zone, nor any acoustic discontinuities with their associated impulsive noise in the far zone.

Of the three source terms in the Ffowcs Williams–Hawkings equation, it is only the monopole term which is determined by the shape and the velocity of the blade itself and so is quasi-static independently of the flow surrounding the blade; the other two source terms are quasi-static only when the ambient flow is quasi-static (i.e. is steady in the rotating frame). Because the source term responsible for impulsive noise is the quadrupole term which is determined by the flow surrounding the tip of the blade, it would in principle be possible to remove the impulsive noise by destroying the quasi-static symmetry of the flow in the transonic region. The characteristics of the noise from a rotating propeller would be altered radically if the propeller could be designed in such a way that the flow in the vicinity of its tip were unsteady and turbulent.

The singularities in the sound field which arise from singularities of sources of sound – such as those arising from the edges of a blade that are considered by Tam (1983), Chapman (1988), Amiet (1988) and De Bernardis & Farassat (1989) – have no bearing on the breakdown of the linearized theory. When a discontinuous source of this kind is, as in §5, replaced by equivalent initial data for the homogeneous wave equation, we face a Cauchy problem for which either the data, or the derivatives of the data, are singular. It is well known, on the other hand, that the singularities of the Cauchy data propagate along characteristics and so give rise to singular solutions. In fact, in cases where characteristics focus and form caustics, the solutions acquire singularities which are of an even higher order than those in the initial data (see Courant & Hilbert 1962, p. 673). The logarithmic singularities in the sound amplitude, which were found in these earlier works, merely reflect the delta-function singularities in the densities of one-dimensional line sources, since the leading and trailing edges of an airfoil effectively act as line sources (De Bernardis & Farassat 1989).

We have analysed one-dimensional line sources of sound, for which s can be expressed as $s_0\,\delta[r-r(\lambda)]\,\delta[z-z(\lambda)]\,\delta[\hat{\varphi}-\hat{\varphi}(\lambda)]$ in terms of a curve parameter λ and a constant s_0, in Appendix C and have shown that when these have a supersonic rotational motion in a direction perpendicular to their own elongation, they give rise to a sound amplitude which is logarithmically divergent on certain surfaces in space. For any given observation point $\boldsymbol{x}_{\mathrm{P}}$ on such a surface, the divergent contribution towards the sound amplitude arises from the element of the line source, $\boldsymbol{x}$, which both lies on the bifurcation surface associated with $\boldsymbol{x}_{\mathrm{P}}$, i.e. approaches $\boldsymbol{x}_{\mathrm{P}}$ with the speed of sound, and radiates in a direction $\boldsymbol{x}-\boldsymbol{x}_{\mathrm{P}}$ perpendicular to the unit vector $\hat{\boldsymbol{t}}$ that is tangent to the line source at $\boldsymbol{x}$, i.e. satisfies the critical condition $\hat{\boldsymbol{t}}\cdot\nabla|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{P}}| = 0$. The shape of the singular surface depends on the shape of the line source. In Appendix C, we have also explicitly specified the singular surfaces associated with straight-line sources which lie along the radial direction in the plane of rotation, or are parallel to the axis of rotation, and have demonstrated that the singularity of the sound amplitude is in fact of a different order (from logarithmic) at any points on these surfaces that lie on the intersection of the source with the sonic cylinder.

To see that these particular singularities are removed when the source is extended, i.e. when the source density is singularity-free, it is only necessary to consider a source distribution for which the density is less singular: the sound amplitude due to a two-dimensional rotating source, which lies in either the meridional ($\hat{\varphi} = \text{const.}$) or the equatorial ($z = \text{const.}$) plane, is itself finite everywhere (Appendix C). As already noted by Ffowcs Williams & Hawkings (1969) in their discussion of shell sources, the integrand in (7) becomes highly singular also when a two-dimensional source moves towards the observer at the wave speed with its normal parallel to the radiation direction, for this condition corresponds to that of tangency of the space–time trajectory of the source with the past light cone of the observer. However, this singularity of the Green's function is integrable in the planar two-dimensional case and only manifests itself in a divergent value for a measurable quantity when the gradient of the sound amplitude is considered at an observation point which is in addition located on the source. In other words, there is a hierarchy of singularities: the sound amplitude due to a point source has an algebraic singularity on the caustic, that due to a line source has a logarithmic singularity on a surface determined by its shape, and that due to a planar shell source is itself finite but has an algebraically singular *gradient* at the sonic cylinder. Only this last singularity persists in the sound field of an extended source – with, of course, a reduced strength (see Appendix A).

Because physically realizable sources, such as airfoils and rotor blades, never have singular densities, the singularities so far discussed in the literature are removed once these sources are modelled more realistically. The singularity which appears in the sound field of an extended source, on the other hand, arises not from singular but from ill-posed Cauchy data: it is a consequence merely of the assumption that the flow is steady in the rotating frame, and hence of the fact that the equivalent initial data have to be prescribed on a space–time hypersurface which is locally characteristic at the sonic cylinder (see §5). There is no way of removing this latter singularity other than by either abandoning the quasi-static constraint (12) or by reformulating the governing equation (1).

Given the quasi-static constraint (12), the singularity discussed in §4 signifies the breakdown of the linearized theory because it is predicted by the solution of (11) for well-behaved source densities and for smooth initial and boundary conditions; that is, because it cannot be attributed to anything other than the inapplicability of the governing field equation. Indeed, had it not been for the fact that (11) is of the mixed type, its breakdown could not have been signalled by the prediction of a singularity: the solutions of a linear hyperbolic partial differential equation, such as the acoustic wave equation in the (r, φ, z, t)-space, for well-behaved source densities and for smooth initial data on space-like hypersurfaces are also smooth themselves. Here, as in all other transonic and transcritical flows (Tuck 1966; Moulden 1984; Cole & Cook 1986), the prediction of a singularity by the linearized theory and the mixed nature of the governing equation are intimately linked (see §5).

I thank A. M. Cargill, C. J. Chapman, F. Farassat and J. E. Ffowcs Williams for their stimulating and helpful comments.

Appendix A

In this appendix we apply the method developed in §5 of Paper I to show that the gradient of the sound amplitude for $\hat{r}_P \to 1+$ diverges like $(\hat{r}_P^2 - 1)^{-\frac{3}{4}}$ in cases – such as those of thickness and loading noise – where the source is distributed over the equatorial plane $z = z_P$. The notation is the same as that in Paper I and the equation numbers belonging to Paper I carry the prefix I.

For an observation point which is located on the source plane, i.e. for $z_P = z$ or $\zeta = 0$, (I 12) and (I 21) yield

$$k_1 = \hat{r}_</\hat{r}_>, \quad \eta = \hat{r}_<^2 - 1, \quad \xi = \hat{r}_<^2(1 - \hat{r}_>^{-2}), \tag{A 1}$$

where $\hat{r}_<(\hat{r}_>)$ is the smaller (larger) of $r_P\omega/c$ and $r\omega/c$. So, from (I 71) and (I 72) we have

$$\Psi_-|_{\zeta=0} \approx \frac{2(\hat{r}_>^2 - 1)^{\frac{1}{4}}}{(\hat{r}_<^2 - 1)^{\frac{3}{4}}[(\hat{r}_>^2 - 1)^{\frac{1}{2}} - (\hat{r}_<^2 - 1)^{\frac{1}{2}}]} \tag{A 2}$$

when both $\hat{r}_< - 1$ and $\hat{r}_> - 1$ are non-negative and much smaller than unity, and $\epsilon_+ \to 0$.

The singularity contribution towards the value of $\partial p/\partial r_P$ from the lower sheet of the bifurcation surface is proportional – as in (I 66) – to the integral of $\Psi_-|_{\zeta=0}$ over the interval $1 \leqslant \hat{r} \leqslant \hat{r}_0$, where $\hat{r}_0$, which plays the role of ξ_0 appearing in (I 75), is a parameter satisfying $0 < \hat{r}_P - 1 < \hat{r}_0 - 1 \ll 1$. However, since the singularity of $\Psi_-|_{\zeta=0}$ at $r = r_P$ is not – like that of Ψ_- at $r = r_P$, $z = z_P$ – integrable, we must here consider the Cauchy principal value of this integral. Breaking up the range of integration into the intervals $1 \leqslant \hat{r} \leqslant \hat{r}_P - \epsilon$ and $\hat{r}_P + \epsilon \leqslant \hat{r} \leqslant \hat{r}_0$ over which the expression in (A 2) has two different forms, performing the integrations and letting ϵ tend to zero, we find that

$$P\int_1^{\hat{r}_0} \Psi_-|_{\zeta=0}\,\mathrm{d}\hat{r} \approx \tfrac{1}{3}\left[\frac{\hat{r}_0^2 - 1}{\hat{r}_P^2 - 1}\right]^{\frac{3}{4}} \tag{A 3}$$

for $0 \leqslant \hat{r}_P - 1 \ll 1$. This must be compared with the corresponding result, (I 79), for a three-dimensional source distribution, which is proportional to $(\hat{r}_P^2 - 1)^{-\frac{1}{4}}$.

The singularity contribution from the upper sheet of the bifurcation surface can be calculated in a similar way: (I A 2) for $\epsilon_- \to 0$ yields

$$\Psi_+|_{\zeta=0} \approx 2\sqrt{3}(\hat{r}_>^2 - 1)^{-\frac{1}{4}}(\hat{r}_<^2 - 1)^{-\frac{1}{2}}[(\hat{r}_>^2 - 1)^{\frac{1}{2}} + (\hat{r}_<^2 - 1)^{\frac{1}{2}}]^{-\frac{1}{2}}, \tag{A 4}$$

which in turn implies that

$$\int_1^{\hat{r}_0} \Psi_+|_{\zeta=0}\,\mathrm{d}\hat{r} \approx 2\sqrt{3}\left[\frac{\hat{r}_0^2-1}{\hat{r}_P^2-1}\right]^{\frac{1}{2}}, \tag{A 5}$$

for $0 \leqslant \hat{r}_P - 1 \ll 1$. Since this is negligible compared to the contribution from the lower sheet in the present case as in §5 of Paper I, it follows that $(\partial\rho/\partial r_P)_{z=z_P}$ diverges like $(\hat{r}_P^2-1)^{-\frac{3}{4}}$ as $\hat{r}_P \to 1+$.

Appendix B

In this appendix we show that the singularity in the gradient of the retarded potential at the sonic cylinder, which in Paper I was derived on the basis of (7), follows also from the Duhamel's form of this potential, (16). Here we use the same notation as that in Paper I and designate the equation numbers belonging to this earlier paper by the prefix I.

Just as (7) of the present paper corresponds to (I 11), the counterpart of (16) is

$$A_\mu(r_P, \hat{\varphi}_P, z_P) = c^{-1}\int_V r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z\, j_\mu(r, z, \hat{\varphi})/R, \tag{B 1}$$

in which $\hat{\varphi}$ stands for the value of $\varphi - \omega t$ at the retarded time:

$$\hat{\varphi} = \hat{\varphi}_P + \varphi - \varphi_P + R\omega/c. \tag{B 2}$$

The Jacobian $\mathrm{d}\hat{\varphi}/\mathrm{d}\varphi$ of the mapping (B 2) vanishes on the bifurcation surface $\phi = \phi_\pm(r,z)$, where $\phi \equiv \hat{\varphi} - \hat{\varphi}_P$ and $\phi_\pm$ are given in (I 22). Let us therefore exclude from V the small volumes $\phi_- - \epsilon_-' < \phi < \phi_- + \epsilon_-$ and $\phi_+ - \epsilon_+ < \phi < \phi_+ + \epsilon_+'$, which enclose the zeros of this Jacobian, by inserting the following combination of step functions in the integrand of equation (B 1):

$$\begin{aligned} K \equiv \theta(-\eta) + \theta(\eta)\{&\theta(\phi - \phi_+ - \epsilon_+') + \theta(\phi_- - \epsilon_-' - \phi) + \theta(\phi_+ - \phi_- - \epsilon_+ - \epsilon_-) \\ &\times [\theta(\phi - \phi_- - \epsilon_-) - \theta(\phi - \phi_+ + \epsilon_+)]\}, \end{aligned} \tag{B 3}$$

and proceed to the limit $\epsilon s \to 0$ after having calculated $\partial A_\mu/\partial r_P$.

Thus the differentiation of (B 1) yields

$$\frac{\partial A_\mu}{\partial r_P} = \frac{1}{c}\lim_{\epsilon s \to 0}\int_V r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z\left[\frac{K}{R}\left(\frac{\omega}{c}\frac{\partial j_\mu}{\partial\hat{\varphi}} - \frac{j_\mu}{R}\right)\frac{\partial R}{\partial r_P} + \frac{j_\mu}{R}\frac{\partial K}{\partial r_P}\right]. \tag{B 4}$$

For the same reasons as those given in §5 of Paper I, the first two terms on the right-hand side of this equation do not receive any contributions from the boundaries of the excluded region in the limit $\epsilon s \to 0$ and need not be considered any further. The third term, on the other hand, only involves delta-functions and so consists entirely of boundary contributions: the derivative of $\theta(\phi - \phi_+ + \epsilon_+')$, for instance, is given by

$$\frac{\partial}{\partial r_P}\theta(\phi - \phi_+ + \epsilon_+') = \left(\frac{\omega}{c}\frac{\partial R}{\partial r_P} - \frac{\partial \phi_+}{\partial r_P}\right)\delta(\phi - \phi_+ - \epsilon_+'). \tag{B 5}$$

In addition, since we are here concerned only with the contribution of the source elements which lie in the vicinity of the observation point, the domain of integration V can be regarded as the image under the mapping $\hat{\varphi} \to \varphi$ of the small volume $\Delta\hat{V}$ (Paper I, §4) of the $(r, z, \hat{\varphi})$-space.

When restricted to such a domain and integrated over φ, the third term in (B 4), which we here denote by $(\partial A_\mu/\partial r_P)_{S'}$, can be written as

$$\left(\frac{\partial A_\mu}{\partial r_P}\right)_{S'} = c^{-1}\bar{j}_\mu \lim_{\epsilon s\to 0}\int_{\Delta\hat{S}} r\,\mathrm{d}r\,\mathrm{d}z\,[\chi_8^+ - \chi_1^- - \theta(\phi_+ - \phi_- - \epsilon_+ - \epsilon_-)(\chi_7^+ - \chi_6^+ + \chi_5^+ - \chi_4^- + \chi_3^- - \chi_2^-)], \quad \text{(B 6)}$$

in which

$$\chi_j^\pm \equiv \frac{1}{R}\left(1+\frac{\omega}{c}\frac{\partial R}{\partial\varphi}\right)^{-1}\left(\frac{\omega}{c}\frac{\partial R}{\partial r_P} - \frac{\partial\phi_\pm}{\partial r_P}\right)\Bigg|_{\varphi=\varphi_P-\pi+2\beta_j}, \quad j = 1,2,\ldots,8, \quad \text{(B 7)}$$

β_j are the angles shown in figure 2 of Paper I, and $\bar{j}_\mu$ is the average value of j_μ over the small volume $\Delta\hat{V}$ (cf. (I 30)).† The derivatives of R in this equation directly follow from (6); the remaining derivatives,

$$\partial\phi_\pm/\partial\hat{r}_P = (r\cos 2\beta + r_P)/R\,|_{\beta=\beta_\pm}, \quad \text{(B 8)}$$

are most easily obtained by differentiating the equation $\phi_\pm = g(\beta_\pm)$.

If we now insert these derivatives in (B 7) and express $\chi_j^\pm$ in terms of the variables introduced in equations (I 12) and (I 37)–(I 44), we arrive at

$$\chi_j^\pm = \frac{\omega^2 k_1^{\frac{1}{4}}}{c^2\hat{r}_P\,\sigma}\,\frac{k_1(\mathrm{sn}^2 u - \mathrm{sn}^2 u_\pm) - \mathrm{cn}\,u\,\mathrm{dn}\,u + \mathrm{cn}\,u_\pm\,\mathrm{dn}\,u_\pm}{(\mathrm{dn}\,u + k_1\,\mathrm{cn}\,u)^2(\mathrm{sn}\,u - \mathrm{sn}\,u_\pm)}\Bigg|_{u=u_j}. \quad \text{(B 9)}$$

For u_3, u_4, u_5 and u_6 whose values in the limits $\epsilon_\pm \to 0$ are u_- or u_+, this expression becomes indeterminate when either ϵ_+ or ϵ_- vanishes. The indeterminacies in (B 9) can easily be removed by writing

$$\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u - \mathrm{cn}\,u_\pm\,\mathrm{dn}\,u_\pm}{\mathrm{sn}\,u - \mathrm{sn}\,u_\pm} = -\frac{(\mathrm{sn}\,u + \mathrm{sn}\,u_\pm)(\mathrm{dn}^2 u + k_1^2\,\mathrm{cn}^2 u_\pm)}{\mathrm{cn}\,u\,\mathrm{dn}\,u + \mathrm{cn}\,u_\pm\,\mathrm{dn}\,u_\pm}, \quad \text{(B 10)}$$

and noting that the right-hand side of this equation has a finite value at $u = u_\pm$. In addition, however, (B 9) becomes singular either when $\epsilon_+ = 0$ and $u_2 \to u_3 \to u_+$ so that $\mathrm{sn}\,u_2 = \mathrm{sn}\,u_3 = \mathrm{sn}\,u_-$ and $\mathrm{dn}\,u_2 = \mathrm{dn}\,u_3 = \mathrm{dn}\,u_-$ but $\mathrm{cn}\,u_2 = \mathrm{cn}\,u_3 = -\mathrm{cn}\,u_-$, or when $\epsilon_- = 0$ and $u_6 \to u_7 \to u_-$ so that $\mathrm{sn}\,u_6 = \mathrm{sn}\,u_7 = \mathrm{sn}\,u_+$ and $\mathrm{dn}\,u_6 = \mathrm{dn}\,u_7 = \mathrm{dn}\,u_+$ but $\mathrm{cn}\,u_6 = \mathrm{cn}\,u_7 = -\mathrm{cn}\,u_+$. In both cases the denominators in (B 9) vanish while the numerators remain finite.

For the same reasons as those given in the case of (B 9), therefore, the only terms in $(\partial A_\mu/\partial r_P)_{S'}$ which can survive in the limit $\epsilon s \to 0$ are the following ones whose integrands possess the above singularities:

$$\left(\frac{\partial A_\mu}{\partial r_P}\right)_{S'_\mp} = \frac{\omega^2}{c^3}\bar{j}_\mu \lim_{\epsilon s\to 0}\int_{\Delta\hat{S}} r\,\mathrm{d}r\,\mathrm{d}z\,\frac{k_1^{\frac{1}{4}}}{\hat{r}_P\,\sigma}\,\Psi'_\mp\,\theta(\phi_+ - \phi_- - \epsilon_+ - \epsilon_-), \quad \text{(B 11)}$$

where

$$\Psi'_- \equiv \frac{\mathrm{cn}\,u_7\,\mathrm{dn}\,u_7 - \mathrm{cn}\,u_+\,\mathrm{dn}\,u_+}{(\mathrm{dn}\,u_7 + k_1\,\mathrm{cn}\,u_7)^2(\mathrm{sn}\,u_7 - \mathrm{sn}\,u_+)} - \frac{\mathrm{cn}\,u_6\,\mathrm{dn}\,u_6 - \mathrm{cn}\,u_+\,\mathrm{dn}\,u_+}{(\mathrm{dn}\,u_6 + k_1\,\mathrm{cn}\,u_6)^2(\mathrm{sn}\,u_6 - \mathrm{sn}\,u_+)}, \quad \text{(B 12)}$$

and

$$\Psi'_+ \equiv \frac{\mathrm{cn}\,u_3\,\mathrm{dn}\,u_3 - \mathrm{cn}\,u_-\,\mathrm{dn}\,u_-}{(\mathrm{dn}\,u_3 + k_1\,\mathrm{cn}\,u_3)^2(\mathrm{sn}\,u_3 - \mathrm{sn}\,u_-)} - \frac{\mathrm{cn}\,u_2\,\mathrm{dn}\,u_2 - \mathrm{cn}\,u_-\,\mathrm{dn}\,u_-}{(\mathrm{dn}\,u_2 + k_1\,\mathrm{cn}\,u_2)^2(\mathrm{sn}\,u_2 - \mathrm{sn}\,u_-)}. \quad \text{(B 13)}$$

† Note that in order to be able to approximate $j_\mu(r,z,\hat{\varphi})$ by $\bar{j}_\mu$, it is essential that we exclude the zeros of the Jacobian $\mathrm{d}\hat{\varphi}/\mathrm{d}\varphi$ by excising an interval in $\hat{\varphi}$, which is one of the arguments of j_μ. We would not have been able to simplify j_μ in this way if we had excluded the zeros of the Jacobian by excising an interval in φ.

The remaining calculations, i.e. the determination of the relevant u_j and the evaluation of the integrals over $\Delta\hat{S}$, are closely related to the calculations outlined in §5 and the Appendix of Paper I. Their end result is that the two quantities $\lim_{\hat{r}_P\to 1}(\partial A_\mu/\partial r_P)_{S'_\pm}$ have precisely the same values as those of $\lim_{\hat{r}_P\to 1}(\partial A_\mu/\partial r_P)_{S_\pm}$ which are given in (I 79) and (I A 7).

Appendix C

In this appendix we consider one-dimensional line sources of sound and show that when these have a supersonic rotational motion in a direction perpendicular to their own elongation, they give rise to a sound amplitude which is logarithmically divergent on certain surfaces in space. To emphasize that the particular singularities discussed in this appendix are removed when the source is extended, we shall also briefly consider a two-dimensional rotating source which lies in the meridional plane. The notation is the same as that in Paper I and the equation numbers belonging to Paper I carry the prefix I.

The first step in the calculation of the sound amplitude from any non-compact source distribution is the determination of the retarded times for fixed $(r, z, \hat{\varphi})$ and $(r_P, z_P, \hat{\varphi}_P)$, i.e. the determination of the solutions β of (I 16). Since we are here interested merely in the singularities of the sound amplitude (which arise from the source elements on the intersection of the source with the bifurcation surface), however, we only need the solutions of (I 16) in the neighbourhoods of source points $(r_\pm, z_\pm, \hat{\varphi}_\pm)$ which lie on one of the sheets $(\pm)$ of the bifurcation surface $\phi = \phi_\pm$ (see (I 22)). So, the transcendental equation (I 16) can be solved, for the purposes of the present appendix, by expanding the function $g(\beta, r, z)$ that appears in this equation into a Taylor series about $\beta = \beta_\pm$, $\hat{r} = \hat{r}_\pm$ and $\hat{z} = \hat{z}_\pm$, where $\hat{r} \equiv r\omega/c$ and $\hat{z} \equiv z\omega/c$.

This Taylor expansion of g, up to the order needed here, is

$$\begin{aligned} g(\beta, r, z) = {} & \phi_\pm + a_{1\pm}(\hat{r}-\hat{r}_\pm) + a_{2\pm}(\hat{z}-\hat{z}_\pm) + a_{3\pm}(\beta-\beta_\pm)^2 \\ & + a_{4\pm}(\hat{r}-\hat{r}_\pm)^2 + a_{5\pm}(\hat{z}-\hat{z}_\pm)^2 + a_{6\pm}(\hat{r}-\hat{r}_\pm)(\beta-\beta_\pm) \\ & + a_{7\pm}(\hat{z}-\hat{z}_\pm)(\beta-\beta_\pm) + a_{8\pm}(\beta-\beta_\pm)^3 + \ldots, \end{aligned} \tag{C 1}$$

in which

$$\left.\begin{aligned} & a_{1\pm} = (\hat{r}_\pm^2 - 1 \pm \Delta)/(\hat{r}_\pm h_\pm), \quad a_{2\pm} = (\hat{z}_\pm - \hat{z}_P)/h_\pm, \\ & a_{3\pm} = \mp 2\Delta/h_\pm, \quad a_{4\pm} = \tfrac{1}{2}(1-a_{1\pm}^2)/h_\pm, \\ & a_{5\pm} = \tfrac{1}{2}(1-a_{2\pm}^2)/h_\pm, \quad a_{6\pm} = -2(\hat{r}_\pm^{-1} - a_{1\pm} h_\pm^{-1}), \\ & a_{7\pm} = 2a_{2\pm}/h_\pm, \quad a_{8\pm} = 4(\tfrac{1}{3} \mp h_\pm^{-2}\Delta), \end{aligned}\right\} \tag{C 2}$$

and

$$\Delta \equiv (\sigma^2 k_1^{-1} - 1)^{\frac{1}{2}} (\sigma^2 k_1 - 1)^{\frac{1}{2}}. \tag{C 3}$$

(The first-order derivative $\partial g/\partial\beta$ vanishes at $\beta = \beta_\pm$, and the mixed second-order and the third-order terms involving $\hat{r}-\hat{r}_\pm$ and $\hat{z}-\hat{z}_\pm$ do not enter the present calculation.) Because the next step in the calculation after solving (I 16) will be to insert the resulting values of β in

$$D = 2\sigma k^{-1}[(1-k^2\sin^2\beta)^{\frac{1}{2}} - \sigma k \sin\beta\cos\beta] \tag{C 4}$$

which appears in the denominator of the Green's function G_0 (see (I 15) and (I 17)), we will also need the following Taylor expansion of D:

$$D(\beta, r, z) = b_{1\pm}(\beta-\beta_\pm) + b_{2\pm}(\hat{r}-\hat{r}_\pm) + b_{3\pm}(\hat{z}-\hat{z}_\pm) + b_{4\pm}(\beta-\beta_\pm)^2 + \ldots. \tag{C 5}$$

where

$$b_{1\pm} = \mp 2\Delta, \quad b_{2\pm} = \tfrac{1}{2}h_\pm a_{6\pm}, \quad b_{3\pm} = a_{2\pm}, \quad b_{4\pm} = 2h_\pm + a_{3\pm} \tag{C 6}$$

(Recall that $\beta_\pm$ and $h_\pm$ are known in terms of $\hat{r}_\pm$ and $\hat{z}_\pm$ from (I 19) and (I 20)).

Let us first consider a one-dimensional source which, in the rotating frame, lies along the straight line $\hat{z} = \hat{z}_0$, $\hat{\varphi} = \hat{\varphi}_0$ perpendicular to the axis of rotation, i.e. which has a density proportional to $\delta(\hat{z}-\hat{z}_0)\,\delta(\hat{\varphi}-\hat{\varphi}_0)$, where $\hat{z}_0$ and $\hat{\varphi}_0$ are constants. In this case, both g and D are functions of the two variables (β, r) only, and (C 1) and (C 5) do not contain any terms involving $\hat{z}-\hat{z}_\pm$. If the observation point is arbitrary, so that all the coefficients $a_{j\pm}$ and $b_{j\pm}$ are non-zero, the dominant terms in (C 1) are the first three existing terms and hence the solution to $g = \phi_\pm$ for $(\hat{r}, \hat{z}_0, \hat{\varphi}_0)$ in the neighbourhoods of the two points $(\hat{r}_{0\pm}, \hat{z}_0, \hat{\varphi}_0)$, at which the line source pierces the two sheets of the bifurcation surface, are given by

$$|\beta-\beta_{0\pm}| = \left|\frac{a_{1\pm}}{a_{3\pm}}(\hat{r}-\hat{r}_{0\pm})+\dots\right|^{\frac{1}{2}}, \tag{C 7}$$

where $\hat{r}_{0+} \leqslant \hat{r} \leqslant \hat{r}_{0-}$, i.e. $(\hat{r}, \hat{z}_0, \hat{\varphi}_0)$ lies inside the bifurcation surface, and there are two solutions differing by sign at each point. Since this and (C 5) imply that D vanishes like $|\hat{r}-\hat{r}_{0\pm}|^{\frac{1}{2}}$ near $(\hat{r}_{0\pm}, \hat{z}_0, \hat{\varphi}_0)$, the corresponding singularity of G_0 is integrable and therefore the sound amplitude is finite at arbitrary observation points. If the observation point is such that the line source crosses the bifurcation surface at a point on the cusp curve, then $\Delta = 0$ and (C 1) and (C 5) yield

$$\beta-\beta_{0\pm} = -\left(\frac{a_{1\pm}}{a_{8\pm}}\right)^{\frac{1}{3}}(\hat{r}-\hat{r}_{0\pm})^{\frac{1}{3}}+\dots, \tag{C 8}$$

and hence a D which vanishes like $(\hat{r}-\hat{r}_{0\pm})^{\frac{2}{3}}$; even in this case the singularity of G_0 is integrable, so long as the observation point does not lie on the source.

However, there are certain observation points outside the above radial line source for which $a_{1-} = 0$ at $(\hat{r}_{0-}, \hat{z}_0, \hat{\varphi}_0)$ and, as a result, the singularity of G_0 is not integrable. (Note that $a_{1+} \neq 0$ for $\hat{r}_\pm > 1$.) For $a_{1-} = 0$, the zeroth and the second-order terms in (C 1) are the dominant ones, and the solutions to $g = \phi_\pm$ are given by

$$\beta-\beta_{0-} = \tfrac{1}{2}[(\hat{r}_{\mathrm{P}}^2-\hat{r}_{0-}^2)^{\frac{1}{2}} \pm (\hat{r}_{\mathrm{P}}^2-2\hat{r}_{0-}^2+1)^{\frac{1}{2}}]^{-1}(\hat{r}-\hat{r}_{0-})+\dots, \tag{C 9}$$

provided that $\hat{r}_{0-}^2 \leqslant \frac{1}{2}(\hat{r}_{\mathrm{P}}^2+1)$. When these are inserted in (C 5), we obtain the following two values for D:

$$D = \mp(\hat{r}_{\mathrm{P}}^2-2\hat{r}_{0-}^2+1)^{\frac{1}{2}}(\hat{r}-\hat{r}_{0-})+\dots, \tag{C 10}$$

both of which give rise to logarithmically divergent contributions towards the sound amplitude.

For any given supersonic element $\hat{r} = \hat{r}_{0-} > 1$ on the radial line source, the constraint $a_{1-} = 0$ is satisfied on the following surface

$$\hat{r}_{\mathrm{P}}^2-(\hat{z}_{\mathrm{P}}-\hat{z}_0)^2/(\hat{r}_{0-}^2-1) = \hat{r}_{0-}^2, \tag{C 11}$$

which represents a hyperboloid in the $(r_{\mathrm{P}}, z_{\mathrm{P}}, \hat{\varphi}_{\mathrm{P}})$-space. When the section $\hat{r}_{\mathrm{P}} \geqslant (2\hat{r}_{0-}^2-1)^{\frac{1}{2}}$ of hyperboloid (C 11) intersects the lower sheet of the caustic associated with the source element $(\hat{r}_{0-}, \hat{z}_0, \hat{\varphi}_0)$, the contribution of this source element towards the sound amplitude diverges at all points on the intersection curve. The sound amplitude of the whole line source is therefore divergent on the following surface which is formed by the collection of such intersection curves:

$$\begin{aligned}\hat{\varphi}_{\mathrm{P}}-\hat{\varphi}_0 = {} & \arcsin\{\tfrac{1}{2}(1-\hat{r}_{\mathrm{P}}^{-2})+[\tfrac{1}{4}(1-\hat{r}_{\mathrm{P}}^{-2})^2-\hat{r}_{\mathrm{P}}^{-4}(\hat{z}_{\mathrm{P}}-\hat{z}_0)^2]^{\frac{1}{2}}\}^{\frac{1}{2}} \\ & -\{\tfrac{1}{2}(\hat{r}_{\mathrm{P}}^2-1)+(\hat{z}_{\mathrm{P}}-\hat{z}_0)^2+[\tfrac{1}{4}(\hat{r}_{\mathrm{P}}^2-1)^2-(\hat{z}_{\mathrm{P}}-\hat{z}_0)^2]^{\frac{1}{2}}\}^{\frac{1}{2}}.\end{aligned} \tag{C 12}$$

Equation (C 12) may be obtained by the elimination of $\hat{r}_{0-}$ between (C 11) and the equation for the caustic associated with $(\hat{r}_{0-}, \hat{z}_0, \hat{\varphi}_0)$, or equivalently the equation for the bifurcation surface associated with $(r_P, z_P, \hat{\varphi}_P)$.

If we set the observation point on the source, i.e. let $\hat{z}_P = \hat{z}_0$ and $\hat{\varphi}_P = \hat{\varphi}_0$, then from (C 11) and (C 12) it follows that the divergence occurs at the sonic cylinder $\hat{r}_P = 1$ and is caused by the source element $\hat{r}_{0-} = 1$ located at the observation point. For $\hat{r}_P = \hat{r}_{0-} = 1$, on the other hand, not only does the first term in the expansion of D that is given in (C 10) vanish – which would of itself imply a stronger singularity in the sound amplitude – but in addition the right-hand side of (C 9), as well as the higher-order terms of D, diverge. This is because for $\hat{r}_P = \hat{r}_{0-} = 1$ we have $h_- = 0$ and the Taylor expansion (C 1) breaks down altogether: g is no longer an analytic function of β at a source point which coincides with the observation point (see Paper I, §4).

The results for a line source parallel to the rotation axis, i.e. for a source density proportional to $\delta(\hat{\varphi} - \hat{\varphi}_0)\,\delta(\hat{r} - \hat{r}_0)$, are very similar to those which are obtained above. In this case, the expansions (C 1) and (C 5) contain terms involving $\beta - \beta_\pm$ and $\hat{z} - \hat{z}_\pm$ only, and for an observation point that renders a_{2+} zero, we have the following counterparts of (C 9) and (C 10):

$$\beta - \beta_{0+} = \pm \frac{\hat{z} - \hat{z}_{0+}}{2(\hat{r}_P^2 - 1)^{\frac{1}{4}}(\hat{r}_0^2 - 1)^{\frac{1}{4}}} + \dots, \tag{C 13}$$

$$D = \mp (\hat{r}_P^2 - 1)^{\frac{1}{4}}(\hat{r}_0^2 - 1)^{\frac{1}{4}}(\hat{z} - \hat{z}_{0+}) + \dots, \tag{C 14}$$

and hence a logarithmic singularity in the sound amplitude. The locus of the singularity in the $(r_P, z_P, \hat{\varphi}_P)$-space is

$$\hat{\varphi}_P - \hat{\varphi}_0 = \arcsin\left[\hat{r}_0^{-1}(1 - \hat{r}_P^{-2})^{\frac{1}{2}} + \hat{r}_P^{-1}(1 - \hat{r}_0^{-2})^{\frac{1}{2}}\right] - (\hat{r}_P^2 - 1)^{\frac{1}{2}} - (\hat{r}_0^2 - 1)^{\frac{1}{2}}. \tag{C 15}$$

This cylindrical surface touches the sonic cylinder along one of its generators, which are straight lines parallel to the z_P-axis. On this generator $\hat{r}_P$ equals 1 and the right-hand side of (C 13) diverges; but this is because the coefficient of $(\beta - \beta_\pm)^2$ in (C 1) vanishes and the term $(\beta - \beta_\pm)^3$ becomes significant in this case. The breakdown of the Taylor expansion itself occurs only if an observation point on the surface (C 15) coincides with a source point, i.e. if $\hat{\varphi}_P = \hat{\varphi}_0$, $\hat{r}_P = \hat{r}_0 = 1$, and hence the source element $\hat{z}_P = \hat{z}_{0+}$ which makes the divergent contribution moves at the speed of sound.

In the case of a general one-dimensional source which is concentrated on an arbitrary curve $r = r(\lambda)$, $z = z(\lambda)$, $\hat{\varphi} = \hat{\varphi}(\lambda)$, expansions (C 1) and (C 5) contain both the terms involving $\hat{r} - \hat{r}_\pm$ and those involving $\hat{z} - \hat{z}_\pm$; all of these, however, can be expressed in terms of $\lambda - \lambda_\pm$, where λ is the curve parameter and $\lambda_\pm$ its value at the intersection of the curve and the bifurcation surface. The coefficient of the first-order term $\lambda - \lambda_\pm$ in expansion (C 1) would vanish if $\partial g/\partial\lambda|_{\lambda=\lambda_\pm} = 0$, i.e. if $\hat{\boldsymbol{t}}\cdot\nabla g = 0$, where $\hat{\boldsymbol{t}}$ is the unit vector tangent to the line source at the point $\lambda = \lambda_\pm$. Since g depends on $\boldsymbol{x}$ only through R (see (I 10)–(I 14)), this is equivalent to the constraint $\hat{\boldsymbol{t}}\cdot\nabla|\boldsymbol{x} - \boldsymbol{x}_P| = 0$ or $\hat{\boldsymbol{t}}\cdot(\boldsymbol{x} - \boldsymbol{x}_P) = 0$. (Note that the conditions $a_{1-} = 0$ and $a_{2+} = 0$ encountered above are particular examples of this constraint in which $\hat{\boldsymbol{t}}$ is the base vector $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_z$, respectively.) As long as certain elements of the source move with the speed of sound in a direction perpendicular to its elongation, therefore, there will be a divergent sound amplitude. Only the shape of the surface on which the sound amplitude diverges is different for different line sources (cf. (C 12) and (C 15)). On the other hand, a supersonic source consisting of a circular arc which rotates along its own elongation in the azimuthal direction does not produce a divergent sound amplitude anywhere except on itself (see (I 36)).

In contrast, the sound amplitude due to a two-dimensional planar source distribution, which rotates about the z-axis, is finite even on the source itself. In the case of a source which lies in the equatorial plane, this follows from the fact that the singularities of the elliptic integrals appearing in (I 36) are integrable. For a two-dimensional rotating source which lies in the meridional plane $\hat{\varphi} = \hat{\varphi}_0$, however, this follows from the present analysis only if the observation point is not located on the sonic cylinder. The point on the bifurcation surface for which the coefficients $a_{1\pm}$ and $a_{2\pm}$ in (C 1) vanish simultaneously, is $\hat{r}_{0\pm} = 1$, $\hat{z}_{0\pm} = \hat{z}_{\mathrm{P}}$. And the solution to $g = \phi_{\pm}$, and the expression for D, in the neighbourhood of this point are

$$\beta - \beta_0 = \pm(\tfrac{3}{2})^{\frac{1}{2}}(\hat{r}-1)^{\frac{1}{2}} - \tfrac{1}{8}(\hat{r}_{\mathrm{P}}^2-1)^{-\frac{1}{2}}(\hat{r}-1)^{-1}(\hat{z}-\hat{z}_{\mathrm{P}})^2 + \dots, \tag{C 16}$$

$$D = 2(\hat{r}_{\mathrm{P}}^2-1)^{\frac{1}{2}}(\hat{r}-1) \mp \tfrac{1}{2}(\tfrac{3}{2})^{\frac{1}{2}}(\hat{r}-1)^{-\frac{1}{2}}(\hat{z}-\hat{z}_{\mathrm{P}})^2 + \dots. \tag{C 17}$$

The corresponding expression for G_0, therefore, has a point singularity at $\hat{r} = 1$, $\hat{z} = \hat{z}_{\mathrm{P}}$ that is integrable, unless $\hat{r}_{\mathrm{P}} = 1$, in which case these Taylor expansions break down altogether. It can be shown – by first integrating the integrand in (I 13) over z and then expanding the argument of the resulting square root in a power series – that the sound amplitude itself is in the case of $\hat{r}_{\mathrm{P}} = 1$, too, finite everywhere. What diverges is the radial gradient of this amplitude at the points where the plane $\hat{\varphi} = \hat{\varphi}_0$ intersects the sonic cylinder (see also Appendix A).

Note added in proof. The following results, which were obtained after the paper was written, are of direct relevance to the discussions in §§4 and 7:
1. It is possible to give a much simpler derivation of the singularity in the gradient of the sound amplitude which occurs at the sonic cylinder by means of a frequency-domain analysis (Ardavan 1991).
2. The sound amplitude of a rigidly rotating two-dimensional shell source can have singularities that are even algebraic, provided that the shape of the shell in the neighbourhood of a point on the sonic cylinder is sufficiently close to the shape of one of the sheets of the observer's bifurcation surface which passes through that point. This follows from an analysis similar to that presented in Appendix C, in which the Taylor expansions of the phase function g and the shape functions $\phi = \phi_{\pm}$ about the point $a_{1\pm} = a_{2\pm} = 0$ are performed in powers of $z - z_{\mathrm{P}}$ and Δ, and are carried as far as the fifth order.

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